

Skorokhod Embeddings In Bounded Time

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Abstract

This article deals with the Skorokhod embedding problem in bounded time for the Brownian motion with drift $X_t = \kappa t + W_t$: Given a probability measure μ we aim at finding a stopping time τ such that the law of X_τ is μ , and τ is almost surely smaller than some given fixed time horizon $T > 0$. We provide necessary and sufficient conditions on the distribution μ for the existence of such bounded stopping times.

Introduction

The problem of finding a stopping time τ such that a Brownian motion stopped at τ has a given centered probability distribution μ , i.e. $\mu \sim W_\tau$, is known as Skorokhod embedding problem (SEP). After its initial formulation by Skorokhod in 1961 ([7]) many solutions have been derived ¹.

This article aims at finding conditions guaranteeing the existence of stopping times τ that are *bounded* by some real number $T < \infty$, and embed a given

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¹For an overview we recommend the survey article by Jan Obloj [5].

distribution in a Brownian motion, possibly with drift. So far this question has not been analyzed in the literature. It is of particular economic interest because it arises naturally in the context of mixed strategy Nash equilibria in dynamic games without observability. To illustrate this point, consider a two player game where both players stop a privately observed Brownian motion. At some fixed time T both players are evaluated and the player who stopped his Brownian motion at a higher value wins a pre-specified prize. A strategy of a player is a stopping time which is bounded by T (see [6]).

The approach of solving the Skorokhod embedding problem presented by Bass in [2], and in more detail in the forthcoming book [3], turns out to be extremely suitable for providing conditions under which a distribution can be embedded in bounded time. Bass's method starts with the representation of a random variable with distribution μ as a stochastic integral with respect to a Brownian motion on the interval $[0, 1]$. Then the time of the representing integral *process* is modified such that the resulting process is a Brownian motion up to some stopping time τ solving the SEP in a weak sense. The time change is governed by an ODE dictating the time speed along every integral path. This allows to derive analytic criteria for the rate of time change to be bounded, and hence the SEP to be solvable in bounded time.

When dealing with a Brownian motion *with drift*, Bass's method can be generalized by appealing to non-linear integral representation as provided by Backward Stochastic Differential Equations (BSDE) (see [1]). The solution process of a BSDE with quadratic growth generator $f(z) = -\kappa z^2$, $\kappa \in \mathbb{R}$, can be time-changed in such a way that one obtains a Brownian motion with drift κ . As in the case without drift this allows, in a first step, to derive a solution of the SEP in a weak sense. Again the link between the time-change and the Brownian paths driving the BSDE is established via an ODE. On the one hand this enables one to solve the SEP in the classical sense, and on the other hand it provides analytic criteria for a distribution to be embeddable in bounded time.

Here is an outline of the following presentation: We start in Section 1 by showing how the small ball asymptotics of a Brownian motion, possibly with drift, entail *necessary* conditions for the SEP to be solvable in bounded time. In Section 2 we give a brief description of Bass's method and its generalization using BSDE techniques. A collection of *necessary* conditions for the SEP to be solvable in bounded time will be given in Section 3. Finally, in Section 4, we briefly describe a contest game with a Nash equilibrium distribution that, appealing to the results of Section 3, can be embedded in a Brownian motion with drift in bounded time.

1 A Necessary Condition on Small Ball Probabilities

Let $(W_t)_{t \in \mathbb{R}_+}$ be a Brownian motion on a probability space (Ω, \mathcal{F}, P) , and denote by (\mathcal{F}_t) the filtration generated by W and extended by the P -null sets of \mathcal{F} .

We denote by $X_t = W_t + \kappa t$, $\kappa \in \mathbb{R}$, the Brownian motion with drift. Let μ be a probability measure on \mathbb{R} , of finite variance $\int x^2 d\mu(x) = v < \infty$ which fulfills the condition

$$\int_{\mathbb{R}} x d\mu(x) = 0 \quad \text{if } \kappa = 0 \quad (1)$$

or

$$\int_{\mathbb{R}} e^{-2\kappa x} d\mu(x) = 1 \quad \text{if } \kappa \neq 0. \quad (2)$$

The conditions (1) and (2) are necessary for μ to be embeddable in bounded time. To prove this observe first that $(W_t)_{t \in \mathbb{R}_+}$ respectively $(e^{-2\kappa X_t})_{t \in \mathbb{R}_+}$ is a martingale. If τ embeds μ in bounded time, then, by Doob's optional stopping theorem, we have $0 = W_0 = \mathbb{E}(W_\tau) = \int_{\mathbb{R}} x d\mu(x)$ and $1 = e^{-2\kappa X_0} = \mathbb{E}(e^{-2\kappa X_\tau}) = \int_{\mathbb{R}} e^{-2\kappa x} d\mu(x)$.

We denote by $F : \mathbb{R} \rightarrow [0, 1]$, $F(x) = \mu((-\infty, x])$, the cumulative distribution function associated with μ . Suppose that a distribution has a hole, i.e. there exists an open interval such that the distribution has mass both to the left *and* to the right of the interval, but no mass on the interval itself. To embed a distribution with a hole, the process X has to enter the no mass interval with some positive probability. Once in the interval, there is a positive probability that the process will not leave it during a finite time period of fixed length. Therefore, distributions with holes can not be embedded with bounded stopping times.

We can develop this argument further. If a distribution can be embedded in bounded time, say before T , then the mass the distribution assigns to each interval can be estimated from below against the probability for the process X to enter the interval and then to stay there up to time T . We can thus derive a necessary condition on the asymptotic small ball probabilities of distributions that are embeddable in bounded time.

Theorem 1. *If a distribution with distribution function F can be embedded before time $T > 0$, then for all $x \in \mathbb{R}$ with $F(x) \notin \{0, 1\}$ it must hold that*

$$\limsup_{\varepsilon \downarrow 0} -\varepsilon^2 \ln(F(x + \varepsilon) - F(x - \varepsilon)) \leq \frac{\pi^2}{8} T. \quad (3)$$

Proof. Throughout we denote by $B_\varepsilon(x)$ the open ball around x of radius $\varepsilon > 0$. The probability for the absolute value of the Brownian motion W to stay within the ball $B_\varepsilon(0)$ up to any time $t \geq 0$ is given by

$$P\left(\sup_{s \in [0, T]} |W_s| < \varepsilon\right) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-\frac{(2n+1)^2 \pi^2}{8\varepsilon^2} T} (-1)^n \quad (4)$$

(see Sec. 5, Ch. X in [4]). From (4) we obtain the following lower bound

$$P\left(\sup_{s \in [0, T]} |W_s| < \varepsilon\right) \geq \frac{4}{\pi} e^{-\frac{\pi^2}{8\varepsilon^2} T}. \quad (5)$$

From (5) we may also derive a lower bound for the absolute value of the Brownian motion *with drift* to stay within $B_\varepsilon(0)$. To this end let Q be the probability measure on \mathcal{F}_T with density $\frac{dQ}{dP} = \mathcal{E}(-\kappa W)_T$. Girsanov's theorem implies that $X_t = W_t + \kappa t$ is a Brownian motion with respect to Q . Thus, using (5) under Q , we obtain

$$\begin{aligned}
P\left(\sup_{s \in [0, T]} |X_s| < \varepsilon\right) &= E^Q \left[e^{\kappa W_T + \frac{\kappa^2}{2} T} \mathbf{1}_{\{\sup_{s \in [0, T]} |X_s| < \varepsilon\}} \right] \\
&= E^Q \left[e^{\kappa X_T - \frac{\kappa^2}{2} T} \mathbf{1}_{\{\sup_{s \in [0, T]} |X_s| < \varepsilon\}} \right] \\
&\geq e^{-\kappa \varepsilon - \frac{\kappa^2}{2} T} Q\left(\sup_{s \in [0, T]} |X_s| < \varepsilon\right) \\
&\geq e^{-\kappa \varepsilon - \frac{\kappa^2}{2} T} \frac{4}{\pi} e^{-\frac{\pi^2}{8\varepsilon^2} T}. \tag{6}
\end{aligned}$$

We proceed by showing (3) for $x \in \mathbb{R}$ for which $F(x) \notin \{0, 1\}$. Assume that there exists a stopping time τ , bounded by $T \in \mathbb{R}_+$, such that X_τ has the distribution F . Denote by $\rho = \inf\{t \geq 0 : X_t = x\}$ the first time the process X hits x . Since $F(x) \notin \{0, 1\}$, the event $A = \{\rho < \tau\}$ occurs with positive probability.

Let \mathcal{F}_ρ be the σ -field generated by ρ and observe that $A \in \mathcal{F}_\rho$. Besides notice that the process $M_h = W_{h+\rho} - W_\rho$, $h \geq 0$, is a Brownian motion independent of \mathcal{F}_ρ .

The mass of F on the ball $B_\varepsilon(x)$ has to be at least as large as the probability that A occurs and that X stays within the ball $B_\varepsilon(x)$ between ρ and T . Therefore

$$\begin{aligned}
F(x + \varepsilon) - F(x - \varepsilon) &\geq P\left(\left\{\sup_{\rho \leq s \leq T} |X_s - X_\rho| < \varepsilon\right\} \cap A\right) \\
&\geq P\left(\left\{\sup_{0 \leq s \leq T} |M_s + \kappa s| < \varepsilon\right\} \cap A\right).
\end{aligned}$$

Since M and \mathcal{F}_ρ are independent, we further obtain that

$$F(x + \varepsilon) - F(x - \varepsilon) \geq P(A) P\left(\sup_{0 \leq s \leq T} |M_s + \kappa s| < \varepsilon\right)$$

Due to (6) we have

$$-\varepsilon^2 \ln[F(x + \varepsilon) - F(x - \varepsilon)] \leq \varepsilon^2 \ln \frac{\pi}{4P(A)} + \varepsilon^3 \kappa + \varepsilon^2 \frac{\kappa^2}{2} T + \frac{\pi^2}{8} T,$$

which implies the result. \square

Theorem 1 entails that a distribution function, embeddable in bounded time, satisfies some simple properties. As usual, we define the inverse of the distribution function F by

$$F^{-1} : [0, 1] \rightarrow [-\infty, \infty], \quad F^{-1}(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\},$$

where $\inf \emptyset = \infty$.

Corollary 1. *If a distribution can be embedded with a bounded stopping time, then its distribution function F is strictly increasing on the closure of $(F^{-1}(0), F^{-1}(1))$ in \mathbb{R} . Moreover, the inverse distribution function F^{-1} is continuous on $(0, 1)$.*

Proof. The first statement is an immediate consequence of Theorem 1. The inverse of a distribution is always right-continuous. Since F is strictly increasing, F^{-1} is also left-continuous, and hence the second statement holds true. \square

We next give an example of a centered distribution on $[-1, 1]$, satisfying the without "holes" assumption, but nevertheless can not be embedded in bounded time.

Example 1. *Let $p > 2$ and $c = \left(\int_{-1}^{+1} e^{-\frac{1}{|x|^p}} dx\right)^{-1}$. Let F be the distribution function of the probability measure on $[-1, 1]$ with density*

$$f(x) = ce^{-\frac{1}{|x|^p}}, \quad x \in [-1, 1].$$

Notice that $F(\varepsilon) - F(-\varepsilon) \leq 2\varepsilon e^{-\frac{1}{\varepsilon^p}}$, $\varepsilon \in (0, 1)$, and hence

$$-\varepsilon^2 \ln(F(\varepsilon) - F(-\varepsilon)) \geq -\varepsilon^2 \ln(2\varepsilon) + |\varepsilon|^{2-p},$$

which converges to ∞ as $\varepsilon \downarrow 0$. Therefore, by Theorem 1, there is no bounded stopping time embedding F .

Using a similar argument we will show in the next proposition that the exponential distribution can not be embedded in bounded time. Intuitively, this is because the tail of the exponential distribution does not fall as fast as the tails of the normal distribution. In the remainder we denote the standard normal distribution function by Φ , and its density function by φ .

Proposition 1. *Let $F(x) = \max\{0, 1 - e^{-\lambda x - 1}\}$ be the centered exponential distribution with parameter $\lambda \in \mathbb{R}_+$. Then F can not be embedded into the Brownian motion $(W_t)_{t \in \mathbb{R}_+}$ in bounded time.*

Proof. Assume τ embeds F in the BM $(W_t)_{t \in \mathbb{R}_+}$ and is almost surely smaller than $T < \infty$. We can use the reflection principle to get that

$$\mathbb{P}(W_\tau \geq y) \leq \mathbb{P}(\sup_{t \leq T} W_t \geq y) = 2\mathbb{P}(W_T \geq y) = 2(1 - \Phi(\frac{y}{\sqrt{T}})).$$

On the one hand we know that if τ embeds F in the Brownian motion, then

$$\mathbb{P}(W_\tau \geq y) = 1 - F(y) = e^{-\lambda y - 1},$$

and hence $1 = \frac{\mathbb{P}(W_\tau \geq y)}{\mathbb{P}(W_\tau \geq y)} \leq \frac{2(1 - \Phi(\frac{y}{\sqrt{T}}))}{e^{-\lambda y - 1}}$ for all $y \geq 0$. On the other hand,

$$\lim_{y \rightarrow \infty} \frac{2(1 - \Phi(\frac{y}{\sqrt{T}}))}{e^{-\lambda y - 1}} = \lim_{y \rightarrow \infty} \frac{2 \frac{1}{\sqrt{T}} \varphi(\frac{y}{\sqrt{T}})}{\lambda e^{-\lambda y - 1}} = \lim_{y \rightarrow \infty} \sqrt{\frac{2}{\pi T \lambda^2}} e^{-\frac{y^2}{2T} + \lambda y + 1} = 0,$$

which is a contradiction due to continuity. \square

2 A short summary of the BSDE approach to the Skorokhod embedding problem

In the following we briefly recall how one can solve the Skorokhod embedding problem with BSDE techniques. We first sketch how to obtain a *weak solution*, i.e. how to construct a Brownian motion and an integrable stopping time such that the drifted BM stopped at this time has a given distribution μ . We then proceed by recalling how the embedding problem can be solved for a given Brownian motion. For a full description we refer to [1].

Weak solutions of the Skorokhod embedding problem

Let μ be a probability measure on \mathbb{R} which is not identical to a Dirac measure. Denote by F the distribution function of μ and define $g = F^{-1} \circ \Phi$. Notice that $\xi = g(W_1)$ is a random variable with distribution μ . Suppose that (Y, Z) are predictable processes satisfying, for all $t \in [0, 1]$, the equation

$$Y_t = \xi - \int_t^1 Z_s dW_s - \kappa \int_t^1 Z_s^2 ds. \quad (7)$$

Equation (7) is usually referred to as a *Backward Stochastic Differential Equation* with quadratic growth generator $f(z) = -\kappa z^2$. Notice that for a pair of processes (Y, Z) to solve (7), it is required that almost surely we have $\int_0^1 Z_s^2 ds < \infty$. The latter condition is necessary for the integrals in (7) to be defined.

Given a solution (Y, Z) of (7) we may define the local martingale $M_t = \int_0^t Z_s dW_s$, $t \in [0, 1]$. Let $\langle M, M \rangle$ be the quadratic variation process of M , and define $S = \langle M, M \rangle_1$. For any $t \in \mathbb{R}_+$ let

$$\tau_t = \begin{cases} \inf\{s \geq 0 : \langle M, M \rangle_s > t\}, & \text{if } t < S, \\ 1, & \text{else.} \end{cases}$$

By the Dambis-Dubins-Schwarz Theorem the time-changed process $B_t = M_{\tau_t}$ is a Brownian motion on $[0, S]$. By extending the probability space if necessary and using a Brownian motion independent of W , one may extend B to a Brownian motion on \mathbb{R}_+ . From (7) we immediately obtain that the Brownian motion B with drift κ , starting in Y_0 and stopped at time S , has the distribution μ . Indeed,

$$B_S + \kappa S + Y_0 = \int_0^1 Z_s dW_s + \kappa \int_0^1 Z_s^2 ds + Y_0 = \xi.$$

We proceed by recalling how one can explicitly construct solutions of the BSDE (7). To this end we will distinguish between the no-drift case $\kappa = 0$, and the case $\kappa \neq 0$.

For $\kappa = 0$, Equation (7) simplifies to the stochastic integral representation of ξ . In particular, if ξ is integrable, then (7) possesses a unique solution and

it must hold $Y_t = E[\xi|\mathcal{F}_t]$. Moreover, the Markov property of W yields that $Y_t = G(t, W_t)$, where

$$G(t, x) = E[g(x + W_1 - W_t)], \quad t \in [0, 1], \quad x \in \mathbb{R}.$$

It is straightforward to show that $G \in \mathcal{C}^{1,2}((0, 1) \times \mathbb{R})$. To simplify notation we use in the following the abbreviation $G_x(t, x) = \frac{\partial G}{\partial x}(t, x)$. Appealing to Ito's formula one can show that the control process Z is given by

$$Z_t = G_x(t, W_t).$$

We next turn to the case $\kappa \neq 0$. For the construction of a solution of (7) in this case, we need to assume that $e^{-2\kappa\xi}$ is integrable. First define $N_t = E[e^{-2\kappa\xi}|\mathcal{F}_t]$ and observe that $N_t = H(t, W_t)$, where

$$H(t, x) = E \left[e^{-2\kappa g(x+W_1-W_t)} \right], \quad t \in [0, 1], \quad x \in \mathbb{R}.$$

One can show that $H \in \mathcal{C}^{1,2}((0, 1) \times \mathbb{R})$ and that $N_t = N_0 + \int_0^t H_x(t, W_s) dW_s$. Straightforward calculations yield that a solution of (7) is given by

$$Y_t = -\frac{1}{2\kappa} \ln H(t, W_t), \quad t \in [0, 1],$$

and

$$Z_t = -\frac{1}{2\kappa} \frac{H_x(t, W_t)}{H(t, W_t)}, \quad t \in (0, 1),$$

(see Lemma 2.2. in [1]). Thus the logarithmic derivative of H , defined by

$$h(t, x) = -\frac{1}{2\kappa} \frac{H_x(t, x)}{H(t, x)}, \quad t \in (0, 1), \quad x \in \mathbb{R},$$

determines the speed of the time change and a fortiori whether a distribution is embeddable in bounded time. Finally, observe that $Y_0 = -\frac{1}{2\kappa} \ln E[e^{-2\kappa\xi}]$. Therefore, if condition (2) is satisfied, then $Y_0 = 0$.

Strong solution

One can establish a functional dependence of the family of stopping times (τ_t) and the Brownian motion B . Indeed, if $\kappa = 0$, then it holds true that

$$\frac{\partial \tau_t}{\partial t} = \frac{1}{G_x^2(\tau_t, G^{-1}(\tau_t, B_t))}, \quad 0 < t < S, \quad (8)$$

(see Proposition 3 in [2]). If $\kappa \neq 0$, then

$$\frac{\partial \tau_t}{\partial t} = \frac{1}{h^2(\tau_t, H^{-1}(\tau_t, \exp(-2\kappa(B_t + \kappa t + Y_0))))}, \quad 0 < t < S, \quad (9)$$

(see Thm 2.3 in [1]). With the functional dependency of the time change on the Brownian paths (8) resp. (9) we may construct an embedding stopping time for *any* Brownian motion. Take for instance the Brownian motion W and define, for all $\omega \in \Omega$, $\sigma_t(\omega)$ as the solution of the differential equation (8) resp. (9) with $B(\omega)$ replaced by $W(\omega)$, and initial condition $\sigma_0(\omega) = 0$. Then, $T = \lim_{t \uparrow 1} \sigma_t^{-1}$ is a stopping time with respect to (\mathcal{F}_t) . Moreover, the pair (T, W) has the same distribution as (S, B) , which implies that the distribution of the stopped process

$$W_T + \kappa T + Y_0$$

is equal to μ .

3 Sufficient criteria for embedding times to be bounded

Throughout this section let μ be a distribution such that (1) resp. (2) is satisfied. Moreover, in view of Corollary 1, it is no restriction to further assume that $g = F^{-1} \circ \Phi$ is strictly increasing.

3.1 Analytical conditions for the solution to be bounded

The stopping time S of the weak solution constructed in Section 2 satisfies, for $\kappa = 0$,

$$S = \int_0^1 G_x^2(t, W_t) dt,$$

and for $\kappa \neq 0$,

$$S = \int_0^1 h^2(t, W_t) dt.$$

Thus the stopping time S , and hence any strong counterpart, is bounded if and only if the functions $x \mapsto \int_0^1 G_x^2(t, x) dt$ resp. $x \mapsto \int_0^1 h^2(t, x) dt$ are bounded in \mathbb{R} . In particular, the distribution μ is embeddable in bounded time if G_x resp. h is bounded.

Suppose that g is absolutely continuous and denote by g' its weak derivative, i.e.

$$g(y) - g(x) = \int_x^y g'(z) dz.$$

Note that in this case we have $G_x(t, x) = \int_{\mathbb{R}} g'(z) \varphi_{1-t}(x - z) dz$. Therefore, if

$$\int_{\mathbb{R}} g'(z) \varphi_{1-t}(x - z) dz \leq \sqrt{T}, \quad x \in \mathbb{R}, \quad t \in [0, 1),$$

then the distribution μ can be embedded before time T in the Brownian motion *without drift*.

Similarly, the derivative of H with respect to x satisfies

$$H_x(x, t) = \int_{\mathbb{R}} -2\kappa g'(z) \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz.$$

Consequently, if we have

$$\frac{\int_{\mathbb{R}} g'(z) \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz}{\int_{\mathbb{R}} \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz} \leq \sqrt{T}, \quad x \in \mathbb{R}, t \in [0, 1),$$

then the distribution μ can be embedded before T in the Brownian motion *with drift*, $B_t = \kappa t + W_t$.

3.2 An easy-to-check Lipschitz condition

The next theorem states that a distribution is embeddable in bounded time if the function $g = F^{-1} \circ \Phi$ is Lipschitz continuous. In particular, the Lipschitz constant provides an upper bound for the stopping time.

Theorem 2. *Suppose that g is Lipschitz continuous with Lipschitz constant \sqrt{T} . Then μ can be embedded in $B_t = W_t + \kappa t$ before T .*

Proof. Because g is Lipschitz continuous it is also absolutely continuous. Due to the Lipschitz continuity of g , we have that the weak derivative satisfies $g' \leq \sqrt{T}$ almost everywhere. We distinguish two cases. In the case of a BM without drift ($\kappa = 0$) we have

$$\int_{\mathbb{R}} g'(z) \varphi_{1-t}(x-z) dz \leq \sqrt{T} \int_{\mathbb{R}} \varphi_{1-t}(x-z) dz = \sqrt{T},$$

and in the case of a BM with drift $\kappa \neq 0$:

$$\frac{\int_{\mathbb{R}} g'(z) \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz}{\int_{\mathbb{R}} \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz} \leq \sqrt{T} \frac{\int_{\mathbb{R}} \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz}{\int_{\mathbb{R}} \exp(-2\kappa g(z)) \varphi_{1-t}(x-z) dz} = \sqrt{T}.$$

□

Notice that g is Lipschitz continuous if and only if the ratio of the densities

$$\frac{\varphi \circ \Phi^{-1}(\alpha)}{f \circ F^{-1}(\alpha)} \tag{10}$$

is bounded at every quantile $\alpha \in [0, 1]$. For every distribution with bounded support this condition is obviously fulfilled if the density f is bounded away from zero.

3.3 Weaker Conditions

For the rest of the section assume that F is a distribution such that the associated function g is absolutely continuous with weak derivative g' . Our next aim is to relax the assumption of Theorem 2 that g' is bounded since it is unnecessarily strong. We will next introduce an integrability condition on g' that guarantees that a distribution can be embedded in bounded time. To simplify the analysis, throughout this subsection we consider only the Brownian motion *without* drift.

Proposition 2. *Let $\kappa = 0$. Suppose that there exists a constant $p > 1$ such that the convolution $(g')^p * \varphi$ is bounded, say by C . Then the distribution F is embeddable with a stopping time bounded by $C^{\frac{2}{p}} p/(p-1)$.*

Proof. Let $q \in (1, \infty)$ be the conjugate of p , i.e. the real number satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Hölder's Inequality yields that

$$\begin{aligned} G_x(t, x) &= \int_{\mathbb{R}} g'(z) \frac{\varphi_{1-t}(x-z)}{\varphi(x-z)} \varphi(x-z) dz \\ &\leq \left(\int_{\mathbb{R}} (g'(z))^p \varphi(x-z) dz \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \frac{\varphi_{1-t}^q(x-z)}{\varphi^{q-1}(x-z)} dz \right)^{\frac{1}{q}} \\ &\leq C^{\frac{1}{p}} \left(\int_{\mathbb{R}} \frac{\varphi_{1-t}^q(x-z)}{\varphi^{q-1}(x-z)} dz \right)^{\frac{1}{q}}. \end{aligned}$$

Note that

$$\begin{aligned} &\int_{\mathbb{R}} \frac{\varphi_{1-t}^q(x-z)}{\varphi^{q-1}(x-z)} dz \\ &= (2\pi(1-t))^{-\frac{q}{2}} (2\pi)^{\frac{q-1}{2}} \int_{\mathbb{R}} \exp\left(-\left(\frac{q}{1-t} \frac{(z-x)^2}{2}\right) + \left((q-1) \frac{(z-x)^2}{2}\right)\right) dz \\ &= (2\pi)^{-\frac{1}{2}} (1-t)^{-\frac{q}{2}} \int_{\mathbb{R}} \exp\left(-\left(\frac{q}{1-t} - (q-1)\right) \frac{(z-x)^2}{2}\right) dz \\ &= (2\pi)^{-\frac{1}{2}} (1-t)^{-\frac{q}{2}} \left(2\pi \frac{1-t}{q - (1-t)(q-1)}\right)^{\frac{1}{2}} \\ &= (1-t)^{-\frac{q-1}{2}} \frac{1}{\sqrt{q - (1-t)(q-1)}}. \end{aligned}$$

This implies

$$\begin{aligned} G_x(t, x) &\leq C^{\frac{1}{p}} (1-t)^{-\frac{q-1}{2q}} (q - (1-t)(q-1))^{-\frac{1}{2q}} \\ &\leq C^{\frac{1}{p}} (1-t)^{-\frac{q-1}{2q}}, \end{aligned}$$

and hence

$$\begin{aligned}
\int_0^1 G_x(t, x)^2 dt &\leq C^{\frac{2}{p}} \int_0^1 (1-t)^{-\frac{q-1}{q}} dt \\
&= C^{\frac{2}{p}} \left(-q(1-t)^{\frac{1}{q}} \Big|_0^1 \right) \\
&= C^{\frac{2}{p}} q.
\end{aligned}$$

□

Corollary 2. *Suppose $\kappa = 0$ and that there exist $q \in (1, \infty]$ such that $\|g'\|_{L^q(\mathbb{R})} < \infty$. Then the distribution is embeddable in bounded time.*

Proof. Let $r \in (1, q)$. Then $(g')^r \in L^{\frac{q}{r}}$. Observe that $\frac{q}{q-r}$ is the conjugate of $\frac{q}{r}$. Young's Inequality for convolutions implies that

$$\|(g')^r * \varphi\|_{\infty} \leq \|(g')^r\|_{\frac{q}{r}} \|\varphi\|_{\frac{q}{q-r}},$$

i.e. $(g')^r * \varphi$ is bounded. The claim follows now from Proposition 2. □

Lemma 1. *Suppose F has compact support $[\underline{x}, \bar{x}]$. Let $q > 1$. If $\int_{\underline{x}}^{\bar{x}} \frac{1}{f^{q-1}(z)} dz$ is finite, then F can be embedded in bounded time.*

Proof. Notice that

$$\begin{aligned}
\int_{-\infty}^{\infty} |g'(z)|^q dz &= \int_{-\infty}^{\infty} \left(\frac{\varphi(z)}{(f \circ F^{-1} \circ \Phi)(z)} \right)^q dz \\
&\leq (2\pi)^{-\frac{q-1}{2}} \int_0^1 \left(\frac{1}{(f \circ F^{-1})(x)} \right)^q dx \\
&= (2\pi)^{-\frac{q-1}{2}} \int_{F^{-1}(0)}^{F^{-1}(1)} \frac{1}{f^{q-1}(z)} dz < \infty,
\end{aligned}$$

and hence the result follows from Corollary 2. □

We next provide an example where $\frac{\varphi \circ \Phi^{-1}}{f \circ F^{-1}}$ is not bounded and consequently g is not Lipschitz continuous, but the assumption of Lemma 1 applies.

Example 2. *Let F be the distribution function with compact support $[-1, 1]$ and density*

$$f(x) = \begin{cases} -x, & \text{for } x \in [-1, 0], \\ x, & \text{for } x \in [0, 1]. \end{cases}$$

Observe that $F(0) = \Phi(0) = 1/2$ and consequently $\frac{(\varphi \circ \Phi^{-1})(1/2)}{(f \circ F^{-1})(1/2)} = \infty$. Moreover, for $p = 1/2$,

$$\int_{\underline{x}}^{\bar{x}} \frac{1}{f^p(z)} dz = \int_{-1}^1 \frac{1}{|x|^p} = 4,$$

which shows that the assumption of Lemma 1 is satisfied.

4 Application to Contest Games

Consider a model in which $n \geq 2$ agents $i \in \{1, 2, \dots, n\}$ face a continuous time stopping problem. Each agent privately observes a Brownian motion $X_t^i = \kappa t + \sigma W_t^i$ with drift κ . For simplicity of the exposition we restrict ourselves to the case $\sigma = 1$ here.

Every agent i chooses a stopping time τ^i with the restriction that whenever $X_t^i = -x_0$ (bankruptcy) he is forced to stop, $x_0 > 0$. The player who stopped at the highest value wins a pre-specified prize (wlog. 1). In case of a tie, the prize is randomly assigned among all players with the highest value.

In [6] it is proven that in every Nash-equilibrium the stopped BM $X_{\tau^i}^i$ is distributed according to

$$F(x) = \sqrt[n-1]{\frac{1}{n} \frac{\exp(-2\kappa(x+x_0)) - 1}{\exp(-2\kappa x_0) - 1}},$$

on the support $[-x_0, F^{-1}(1)]$. In the following paragraph we will see that the Lipschitz condition of Theorem 2 applies to the equilibrium distribution F . The density $f = F'$ is given by

$$\begin{aligned} f(x) &= \frac{1}{n-1} \left(\frac{1}{n} \frac{\exp(-2\kappa(x+x_0)) - 1}{\exp(-2\kappa x_0) - 1} \right)^{\frac{1}{n-1}-1} \frac{1}{n} (-2\kappa) \frac{\exp(-2\kappa(x+x_0))}{\exp(-2\kappa x_0) - 1} \\ &= \frac{1}{n-1} F(x)^{-n+2} 2|\kappa| \left| F(x)^{n-1} + \frac{1}{n} \frac{1}{\exp(-2\kappa x_0) - 1} \right| \\ &\geq \frac{1}{(n-1)n} 2|\kappa| \left| \frac{1}{\exp(-2\kappa x_0) - 1} \right|. \end{aligned}$$

Because for all $x \in \mathbb{R}$

$$g'(x) = \frac{\varphi(x)}{(f \circ F^{-1} \circ \Phi)(x)} \leq \sup_{z \in [-x_0, F^{-1}(1)]} \frac{1}{f(z)} = n(n-1) \frac{|\exp(-2\kappa x_0) - 1|}{2|\kappa|},$$

it follows that F can be embedded in bounded time. This property is crucial for the economic interpretation of the results as in real world applications contests are always bounded in time.

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