OPTIMAL STOPPING WITH PRIVATE INFORMATION

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ABSTRACT. Many economic situations are modeled as stopping problems. Examples are timing of market entry decisions, search, and irreversible investment. We analyze a principal-agent problem where the principal and the agent have different preferences over stopping rules. The agent privately observes a signal that influences his own and the principal’s payoff. Based on his observation the agent decides when to stop. In order to influence the agents stopping decision the principal chooses a transfer that conditions only on the time the agent stopped.

We derive a monotonicity condition such that all cut-off stopping rules can be implemented using such a transfer. The condition generalizes the single-crossing condition from static mechanism design to optimal stopping problems. We give conditions under which the transfer is unique and derive a closed form solution based on reflected processes. We prove that there always exists a cut-off rule that is optimal for the principal and, thus, can be implemented using a posted-price mechanism.

An application of our result leads to a new purely stochastic representation formula for the value function in optimal stopping problems based on reflected processes. Finally, we apply the model to characterize the set of implementable policies in the context of job search and irreversible investment.

Keywords: Dynamic Mechanism Design, Optimal Stopping, Dynamic Implementability, Posted-Price Mechanism

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1. Introduction

Many economic situations are modeled as stopping problems. Examples are job search, timing of market entry decisions, or irreversible investment decisions. In an optimal stopping problem information arrives over time and a decision maker decides at every point in time whether to stop or continue. The stopping decision is irreversible and, thus, stopping today implies forgoing the option to stop later with a potentially larger return.

The literature on optimal stopping assumes that the decision maker observes all relevant information directly. There are, however, many economic environments modeled as optimal stopping problems where private information is a crucial feature. Important examples include the regulation of a firm that decides when to enter a market, or the design of unemployment benefits when the worker privately observes his job offers. Consider for example a pharmaceutical firm that observes the stochastically changing demand for a new product and needs to decide on whether and when to enter the market for this product. If the firm starts selling the product, it pays an investment cost and receives the flow returns from selling the product at all future times. When timing its market entry decision, the firm solves an irreversible investment problem. As famously shown in the irreversible investment literature, it is optimal for the firm to delay investment beyond the point where the expected return of investing becomes positive (for an excellent introduction in the irreversible investment literature see Dixit and Pindyck [2008]). While this investment strategy is optimal for the firm, it is typically wasteful from a social perspective: As the firm does not internalize the consumer surplus, it will enter the market too late. A question that naturally arises is how a regulator can improve the incentives of the firm if the demand is private information of the firm. The result we derive in this paper implies that the regulator can always implement a socially efficient market entry decision through a simple posted-price-like-mechanisms that conditions on when the firm enters.

Our model is a dynamic principal-agent model. The agent privately observes a randomly changing signal and chooses a stopping rule. The principal observes the stopping decision of the agent, but not the signal. The principal commits to a transfer in order to influence the agent’s decision. Our main result is that under a single-crossing condition all cut-off rules can be implemented using transfers that condition only on the time the agent stopped. As the transfer does not depend on private information of the agent it can be implemented without communication. This feature resembles a posted-price mechanism and makes our mechanism especially appealing from an applied perspective. Finally, we show that the minimal stopping rule that is optimal for the principal is a cut-off rule and thus can be implemented.
We establish our result as follows. First, we derive a dynamic single-crossing condition that ensures the monotonicity of the continuation gain in the agent’s signal at every point in time. A consequence of this monotonicity is that the incentive to stop is increasing in the type. Thus, to implement a cut-off rule it suffices to provide incentives to the (marginal) cut-off type. Intuitively, taking future transfers as given, the transfer providing incentives to the marginal type today could be calculated recursively. But as future transfers are endogenous, this recursive approach is difficult and requires the calculation of a value function at every point in time. We circumvent all these problems attached to the recursive approach by directly constructing transfers using reflected stochastic processes. We define a reflected process as a Markov process that equals the original process as long as the latter stays below the cut-off. Once the original process exceeds the cut-off, the reflected process follows its own dynamics and is defined to stay below the cut-off.

We show that every cut-off rule can be implemented through a transfer that equals the agent’s expected payoff evaluated at the process reflected at the cut-off. To the best of our knowledge, we are the first to use reflected processes in the context of mechanism design. We also believe that this paper contributes to the mathematical literature on optimal stopping as we are able to give a new, completely probabilistic sufficient condition for optimal stopping rules.

The approach of this paper differs from the standard mechanism-design one, since we do not rely on the revelation principle and direct mechanisms. We do not use direct mechanisms in order to bypass technical problems that a formalization of the cheap talk protocol between the principal and the agent would entail: the agent would have to communicate with the principal constantly and, hence, the space of communication strategies would be very rich. As optimal communication strategies are not necessarily Markovian, an optimization over those is a hard problem. The direct mechanism design approach, nevertheless, has been successfully used in general discrete-time settings in Bergemann and Välimäki (2010) for welfare maximization, in Pavan et al. (2012) for revenue maximization, or in continuous time to solve principal-agent problems in Sannikov (2008) and Williams (2011).

Surprisingly, it turns out that for optimal stopping problems the focus on posted-price mechanisms is not restrictive from a welfare perspective. We show that a principal-optimal full information stopping rule (first best) is always implementable by posted-price mechanisms. Moreover, while direct mechanisms often require communication and a transfer at every point in time, our posted-price mechanism demands no communication and only a single transfer at the time of stopping.

The paper proceeds as follows. Section 2 introduces the model. In Section 3 we show that all cut-off rules are implementable using posted-price mechanisms and derive the transfer. Furthermore Section 3 establishes that the minimal stopping rule that is optimal for the
principal is a cut-off rule. Section 4 presents applications to irreversible investment, search, and proves that the classical static implementability result is a special case of our dynamic result. Section 5 concludes.

2. The Model

We first give an informal presentation of the model before we present all the precise mathematical assumptions. There is a risk-neutral principal and a risk-neutral agent, who depending on the stopping rule $\tau$ and the signal receive $X$ a payoff of

\[
\text{agent: } V(\tau) = \mathbb{E} \left[ \int_0^\tau h(t, X_t)dt + g(\tau, X_\tau) \right] \\
\text{principal: } W(\tau) = \mathbb{E} \left[ \int_0^\tau \beta(t, X_t)dt + \alpha(\tau, X_\tau) \right].
\]

The signal $X$ is a Markov process which is only observed by the agent. Depending on his past observation of the signal the agent can decide at every point in time between stopping and continuing. In order to influence the agents decision the principal commits to a transfer $\pi$ that only conditions on the time the agent stopped. The transfer $\pi$ implements the stopping rule $\tau^*$ if $\tau^*$ is optimal for the agent given he receives the transfer $\pi$ in addition to his payoffs $g$ and $h$

\[
V(\tau^*) + \pi(\tau^*) \geq V(\tau) + \pi(\tau) \quad \text{for all } \tau.
\]

The next sections state the precise assumptions of the model.

2.1. Evolution of the Private Signal. Time is continuous and indexed by $t \in [0, T]$ for some fixed time horizon $T < \infty$. At every point in time $t$ the agent privately observes a real valued signal $X_t$. The signal is adapted to the filtration $\mathcal{F}$ of some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]})$. The filtered probability space supports a one-dimensional $\mathcal{F}$-Brownian motion $W_t$ and a $\mathcal{F}$-Poisson random measure $\tilde{N}(dt dz)$ with compensator $\nu(dz)$ on $\mathbb{R}$. We denote by $\tilde{N}(dt dz) = N(dt dz) - \nu(dz)vt$ the compensated Poisson random measure.

The time zero signal $X_0$ is distributed according to the cumulative distribution function $F : \mathbb{R} \to [0, 1]$. The signal observed by the agent $X_t$ changes over time and follows the inhomogeneous Markovian jump diffusion dynamics

\[
dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + \int_\mathbb{R} \gamma(t, X_{t-}, z)\tilde{N}(dtdz).
\]

The functions $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the usual assumptions listed in Appendix A to ensure that the process exists and is well-behaved. Under these assumptions there exists a path-wise unique solution $(X^{t,x}_s)_{s \geq t}$ to (2.1) for every initial condition $X^{t,x}_t = x$.

\[\text{We show later that discrete time models are a special case where we restrict the set of possible strategies.} \]
We assume that jumps preserve the order of signals. At every point in time \( t \) a lower signal \( x \leq x' \) before a jump has a lower level after a jump

\[
x + \gamma(t, x, z) \leq x' + \gamma(t, x', z) \quad \nu(dz) - a.s.
\]

This implies a comparison result (c.f. Peng and Zhu 2006): The path of a signal starting at a lower level \( x \leq x' \) at time \( t \) is smaller than the path of a signal starting in \( x' \) at all later times \( s > t \)

\[
X_{s}^{t,x} \leq X_{s}^{t,x'} \quad \mathbb{P} - a.s.
\]

2.2. Information, Strategies and Payoffs of the Agent. Based on his observations of the signal the agent decides when to stop. The agent can only stop at times \( T \subseteq [0, T] \).

Note that we can embed discrete time in our model by only allowing the agent to stop at a finite number of points \( T = \{1, 2, \ldots, T\} \). Denote by \( \mathcal{T} \) the set of \( \mathcal{F} \) adapted stopping rules taking values in \( T \). A stopping rule is a complete contingent plan which maps every history observed by the agent \((t, (X_s)_{s \leq t})\) into a binary stopping decision. The agent is risk-neutral and receives a flow payoff \( h : [0, T] \times \mathbb{R} \to \mathbb{R} \) and terminal payoff \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \).

His value \( V : \mathcal{T} \to \mathbb{R} \) of using the stopping rule \( \tau \) equals

\[
V(\tau) = \mathbb{E} \left[ \int_{0}^{\tau} h(t, X_t) dt + g(\tau, X_{\tau}) \right].
\]

The problem analyzed in the optimization theory literature is to find an agent-optimal stopping rule, i.e. a stopping rule \( \tau^* \) such that for all stopping rules \( \tau \in \mathcal{T} \)

\[
V(\tau^*) \geq V(\tau).
\]

To ensure that the optimization problem is well posed we assume that \( g \) and \( h \) are Lipschitz continuous and of polynomial growth in the \( x \) variable. Moreover we assume that \( g \) is twice differentiable in the \( x \) variable and once in the \( t \) variable. Then the infinitesimal

\[
(\mathcal{L}g)(t, x) = \lim_{\Delta \searrow 0} \frac{1}{\Delta} (\mathbb{E} [g(t + \Delta, X_{t+\Delta})|X_t = x] - g(t, x))
\]

is well defined for every \( t \) and every \( x \) and we have the identity

\[
(\mathcal{L}g)(t, x) = g_t(t, x) + \mu(t, x)g_x(t, x) + \frac{1}{2}\sigma(t, x)^2 g_{xx}(t, x)
\]

\[
+ \int_{\mathbb{R}} g(t, x + \gamma(t, x, z)) - g(t, x) - g_x(t, x)\gamma(t, x, z)\nu(dz).
\]

Finally we assume that the payoffs of the agent satisfy the following single-crossing condition

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Note that our model can easily extended to include a flow payoff after stopping, i.e. payoffs of the form

\[
\mathbb{E} \left[ \int_{0}^{\tau} h(t, X_t) dt + g(\tau, X_{\tau}) + \int_{\tau}^{T} \tilde{h}(t, X_t) dt \right].
\]
Definition 1 (Dynamic Single-Crossing). The mapping \( x \mapsto h(t,x) + (\mathcal{L}g)(t,x) \) is non-increasing.

The dynamic single-crossing condition means that if after every history \( \mathcal{F}_t \) the expected change in the payoff of the agent if he waits an infinitesimal short time is decreasing in the current signal \( X_t = x \)

\[
\lim_{\Delta \searrow 0} \frac{1}{\Delta} \mathbb{E} \left[ \int_0^{t+\Delta} h(s,X_s)ds + g(t,\Delta, X_{t+\Delta}) - \int_0^t h(s,X_s)ds + g(t,X_t) \mid \mathcal{F}_t \right] \\
= \lim_{\Delta \searrow 0} \frac{1}{\Delta} \mathbb{E} \left[ \int_t^{t+\Delta} h(s,X_s)ds + g(t,\Delta, X_{t+\Delta}) - g(t,X_t) \mid X_t = x \right] \\
= h(t,x) + (\mathcal{L}g)(t,x).
\]

The dynamic single-crossing condition is a condition naturally satisfied in most optimal stopping problems. Examples in the economic literature are irreversible investment (Dixit and Pindyck [2008]), learning in one armed bandit models, job and other search problems as well as American options.

2.3. Information, Strategies and Payoffs of the Principal. Similar to the agent the principal is risk neutral and wants to maximize the functional \( W : \mathcal{T} \to \mathbb{R} \) which is the sum of a flow payoff \( \beta : [0,T] \times \mathbb{R} \to \mathbb{R} \) and terminal payoff \( \alpha : [0,T] \times \mathbb{R} \to \mathbb{R} \)

\[
W(\tau) = \mathbb{E} \left[ \int_0^\tau \beta(t,X_t)dt + \alpha(\tau,X_\tau) \right].
\]

The payoffs \( \alpha, \beta \) of the principal satisfy the same regularity assumptions as the payoffs of the agent \( g, h \) as well as the dynamic single-crossing condition. We call a stopping time \( \tau^* \) principal-optimal if it maximizes the utility of the principal, \( W(\tau^*) \geq W(\tau) \) for all \( \tau \in \mathcal{T} \). Note that the principal optimal stopping time (first best) is the stopping time the principal would like to use if he observes the signal \( X \).

The principal however, does not observe the signal \( X \). He only knows the distribution \( F \) of \( X_0 \) and the probability measure \( \mathbb{P} \) describing the evolution of \( X \), i.e \( \mu, \sigma \) and \( \gamma \). As the signal \( X \) is unobservable the principal has no information about the realized flow payoffs of the agent or the agents expectation about future payoffs. The only information the principal observes is the realized stopping decision of the agent. In order to influence this decision she commits to a transfer scheme \( \pi : [0,T] \to \mathbb{R} \) that depends only on the time the agent stopped. As the transfer only conditions on publicly observable information it requires no communication between the principal and the agent and can be interpreted as a posted-price. We define the set of stopping rules that are implementable through a transfer.
Definition 2 (Implementability). A stopping rule \( \tau^* \in \mathcal{T} \) is implemented by the transfer \( \pi \) if for all stopping rules \( \tau \in \mathcal{T} \)

\[
\mathbb{E} \left[ \int_0^{\tau^*} h(t, X_t) dt + g(\tau^*, X_{\tau^*}) + \pi(\tau^*) \right] \geq \mathbb{E} \left[ \int_0^{\tau} h(t, X_t) dt + g(\tau, X_\tau) + \pi(\tau) \right].
\]

A transfer \( \pi \) implements the stopping rule \( \tau^* \) if the stopping rule \( \tau^* \) is optimal for the agent if he receives the transfer \( \pi \) in addition to his payoffs \( h \) and \( g \). Implementability generalizes the notion of optimality for stopping rules, as a stopping rule is optimal if and only if it is implemented by a constant transfer.\(^3\)

3. Implementable Stopping Rules

We introduce the notion of cut-off rules. Cut-off rules are stopping rules such that the agent stops the first time the signal \( X_t \) exceeds a time-dependent threshold \( b \).

Definition 3 (Cut-Off Rule). A stopping rule \( \tau \in \mathcal{T} \) is called a cut-off rule if there exists a function \( b : [0, T] \rightarrow \mathbb{R} \) such that

\[
\tau = \inf \{ t \geq 0 \mid X_t \geq b(t) \}.
\]

In this case we write \( \tau = \tau_b \). If \( b \) is càdlàg we call \( \tau_b \) a regular cut-off rule.\(^4\)

In Section 3.2 we show that all regular cut-off rules are implementable. The associated transfer admits an explicit representation in terms of the reflected version of \( X \) which we introduce in Section 3.1. Under the assumption, that the agent stops the first time it is optimal to stop we show in Section 3.3 that every implementable stopping rule, is a cut-off rule.

3.1. Reflected Processes. For a given barrier \( b : [0, T] \rightarrow \mathbb{R} \) the reflected version of \( X \) is a process \( \tilde{X} \) which evolves according to the same dynamics as \( X \) as long it is strictly below \( b \) but is pushed downwards any time it tries to exceed \( b \).

Definition (Reflected Process). Let \( b : [0, T] \rightarrow \mathbb{R} \) be a càdlàg function. A processes \((\tilde{X}_t)_{0 \leq t \leq T}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) is called a reflected version of \( X \) with barrier \( b \) if it satisfies the conditions

1) \( \tilde{X} \) is constrained to stay below \( b \): For every \( t \in [0, T] \) we have \( \tilde{X}_t \leq b(t) \) a.s.

2) Until \( X \) hits the barrier both processes coincide: For every \( 0 \leq t < \tau_b \) we have \( X_t = \tilde{X}_t \) a.s.

\(^3\)The definition of implementability does not require a regularity assumption on the transfer \( \pi \) ensuring that the resulting optimization problem of the agent is well posed, as all transfers that lead to an ill-posed optimization problem implement no stopping rule and thus are irrelevant.

\(^4\)A function \( b : [0, T] \rightarrow \mathbb{R} \) is called càdlàg if it is right-continuous and has left-limits.
Figure 3.1. Realization of a random walk (left) and a Brownian motion (right) starting in $X_0 = 8$ reflected at the constant barrier $b(t) = 10$.

3) $\tilde{X}$ is always smaller than $X$: For every $t \in [0,T]$ we have $\tilde{X}_t \leq X_t$ a.s.
4) When $X$ hits the barrier, $\tilde{X}$ is at $b$: $\tilde{X}_{\tau_b} = b(\tau_b)$ a.s.

Moreover, we assume that for every stopping rule $\tau \in \mathcal{T}$ and every $\mathcal{F}_\tau$ measurable random variable $\eta$ there exists a process $(\tilde{X}^{\tau,\eta}_t)_{t \leq T}$ such that

5) $\tilde{X}^{\tau,\eta}$ starts at time $\tau$ in $\eta$: $\mathbb{P}[\tilde{X}^{\tau,\eta}_\tau = \eta] = 1$.
6) $\tilde{X}$ has the strong Markov Property: We have $\tilde{X}^{\tau,\tilde{X}_\tau}_t = \tilde{X}_t$ a.s. for every $t \in [\tau,T]$.
7) A higher value $x' \geq x$ of $\tilde{X}$ at time $t$ leads to higher values of $\tilde{X}$ at all later times $s \geq t$: $\tilde{X}^{\tau,x}_s \leq \tilde{X}^{\tau,x'}_s$ a.s.

The next Proposition ensures the existence of a reflected version of $X$ for a wide range of settings.

**Proposition 4.** Let $b$ be a càdlàg function.

a) If $X$ has no jumps (i.e. $\gamma = 0$), then there exists a version of $X$ reflected at $b$.

b) If $X$ is non decreasing and $b$ non increasing, then a version of $X$ reflected at $b$ is given by $\tilde{X}^{\tau,\eta}_s = \min\{X^{\tau,\eta}_s,b(s)\}$.

Proposition 4 shows that reflected versions of the process exist for very large classes of environments, especially Diffusion processes. One assumption of Proposition 4 will be satisfied in all the examples we discuss later.

3.2. All Cut-Off Rules are implementable. In this section we prove that every cut-off rule is implementable. For a given cut-off rule we define the transfer and explicitly verify that it implements the cut-off rule. The transfer equals the expected future change in payoff evaluated at the process reflected at the cut-off.
**Theorem 5.** Let $b : [0, T] \to \mathbb{R}$ such that there exists a version $\tilde{X}$ of $X$ reflected at $b$. Then $\tau_b$ is implementable by the transfer

$$
\pi(t) = \mathbb{E} \left[ \int_t^T h(s, \tilde{X}_{s,b}^t) + (\mathcal{L}g)(s, \tilde{X}_{s,b}^t) \, ds \right].
$$

**Proof.** Let $\pi$ be given by Equation (3.1). To shorten notation we define $z(t, x) = h(t, x) + (\mathcal{L}g)(t, x)$. By the single-crossing condition $x \mapsto z(t, x)$ is non-increasing. We need to show for all stopping times $\tau \in \mathcal{T}$

$$
\mathbb{E} \left[ \int_0^{T_{\tau}} h(t, X_t) dt + g(\tau, X_\tau) + \pi(\tau) \right] \leq \mathbb{E} \left[ \int_0^{\tau_b} h(t, X_t) dt + g(\tau_b, X_{\tau_b}) ds + \pi(\tau_b) \right].
$$

As $X$ is a Feller process and the generator of $g$ is well defined we can apply Dynkin’s formula to get that (3.2) is equivalent to

$$
\mathbb{E} \left[ \int_0^{T_{\tau}} z(s, X_s) ds + \pi(\tau) \right] \leq \mathbb{E} \left[ \int_0^{\tau_b} z(s, X_s) ds + \pi(\tau_b) \right].
$$

First, we prove that the stopping min$\{\tau, \tau_b\}$ performs at least as well as $\tau$. It follows from the strong Markov property of $\tilde{X}$ that

$$
\pi(\tau) = \mathbb{E} \left[ \int_T^T z(s, \tilde{X}_{s,b}^t) ds \mid t = \tau \right] = \mathbb{E} \left[ \int_{\tau}^T z(s, \tilde{X}_{s,b}^{\tau,b(\tau)}) ds \mid \mathcal{F}_{\tau} \right].
$$

This allows us to compute

$$
\mathbb{E} \left[ 1_{\{\tau_b < \tau\}} \pi(\tau) \right] = \mathbb{E} \left[ 1_{\{\tau_b < \tau\}} \mathbb{E} \left[ \int_{\tau}^T z(s, \tilde{X}_{s,b}^{\tau,b(\tau)}) ds \mid \mathcal{F}_{\tau} \right] \right] = \mathbb{E} \left[ \mathbb{E} \left[ 1_{\{\tau_b < \tau\}} \int_{\tau}^T z(s, \tilde{X}_{s,b}^{\tau,b(\tau)}) ds \mid \mathcal{F}_{\tau} \right] \right] = \mathbb{E} \left[ 1_{\{\tau_b < \tau\}} \int_{\tau}^T z(s, \tilde{X}_{s,b}^{\tau,b(\tau)}) ds \right]
$$

Where the second last step follows as $\{\tau_b < \tau\}$ is $\mathcal{F}_{\tau}$ measurable and the last step applies the law of iterated expectations. We compute

$$
\mathbb{E} \left[ 1_{\{\tau_b < \tau\}} \left( \int_0^{T_{\tau}} z(s, X_s) ds + \pi(\tau) \right) \right] = \mathbb{E} \left[ 1_{\{\tau_b < \tau\}} \left( \int_0^{\tau_b} z(s, X_s) ds + \int_{\tau_b}^{T_{\tau}} z(s, X_s) ds + \int_{\tau}^{T_{\tau}} z(s, \tilde{X}_{s,b}^{\tau,b(\tau)}) ds \right) \right].
$$

The comparison principle for the reflected process $\tilde{X}_{\tau_b,b(\tau)}$ with the original process $X$ yields for every time $s \geq \tau_b$ after the process hit the barrier that the reflected process is not greater than the original process $X_{\tau_b,b(\tau)}$. As the reflected process $\tilde{X}$ is Markovian it follows that $\tilde{X}_{\tau_b,b(\tau)} = \tilde{X}_{\tau_b,b(\tau)}$ for all times $s \geq \tau > \tau_b$. By the comparison principle for reflected processes we have that for all times $s \geq \tau > \tau_b$ the reflected process started in $(\tau, b(\tau))$ is not
smaller than the reflected process started in \((\tau, \tilde{X}_{\tau}^{\tau_b, b(\tau)})\), i.e. \(\tilde{X}_{\tau}^{\tau, b(\tau)} \geq \tilde{X}_{\tau}^{\tau_b, b(\tau_b)}\). These two inequalities combined with the monotonicity of \(z\) yield that for all \(\tau_b < \tau\)

\[
\int_{\tau_b}^{\tau} z(s, X_s)ds \leq \int_{\tau_b}^{\tau} z(s, \tilde{X}_s^{\tau_b, b(\tau_b)})ds
\]

\[
\int_{\tau}^{T} z(s, \tilde{X}_s^{\tau, b(\tau)})ds \leq \int_{\tau}^{T} z(s, \tilde{X}_s^{\tau_b, b(\tau_b)})ds
\]

Hence conditional on \(\tau_b < \tau\) the stopping rule \(\tau_b\) gives the agent a higher expected payoff than the stopping rule \(\tau\)

\[
\mathbb{E}\left[1_{\{\tau_b < \tau\}} \left(\int_{0}^{\tau} z(s, X_s)ds + \pi(\tau)\right)\right]
\]

\[
\leq \mathbb{E}\left[1_{\{\tau_b < \tau\}} \left(\int_{0}^{\tau_b} z(s, X_s)ds + \int_{\tau_b}^{\tau} z(s, \tilde{X}_s^{\tau_b, b(\tau_b)})ds\right) + \int_{\tau}^{T} z(s, \tilde{X}_s^{\tau_b, b(\tau_b)})ds\right]
\]

\[
= \mathbb{E}\left[1_{\{\tau_b < \tau\}} \left(\int_{0}^{\tau_b} z(s, X_s)ds + \pi(\tau_b)\right)\right]
\]

Consequently it is never optimal to continue after \(\tau_b\). The agent is at least as well of if he uses the stopping rule \(\min\{\tau, \tau_b\}\) instead of \(\tau\)

\[
\mathbb{E}\left[\int_{0}^{\tau} z(s, X_s)ds + \pi(\tau)\right]
\]

\[
= \mathbb{E}\left[1_{\{\tau_b < \tau\}} \left(\int_{0}^{\tau} z(s, X_s)ds + \pi(\tau)\right)\right] + \mathbb{E}\left[1_{\{\tau_b \geq \tau\}} \left(\int_{0}^{\tau} z(s, X_s)ds + \pi(\tau)\right)\right]
\]

\[
\leq \mathbb{E}\left[\int_{0}^{\tau_b \wedge \tau} z(s, X_s)ds + \pi(\tau \wedge \tau_b)\right].
\]

Thus it suffices to consider stopping rules \(\tau \leq \tau_b\). We have

\[
\mathbb{E}\left[\int_{0}^{\tau} z(s, X_s)ds + \pi(\tau)\right]
\]

\[
= \mathbb{E}\left[\int_{0}^{\tau} z(s, X_s)ds + \int_{\tau}^{\tau_b} z(s, \tilde{X}_s^{\tau, b(\tau)})ds + \int_{\tau_b}^{T} z(s, \tilde{X}_s^{\tau_b, b(\tau)})ds\right].
\]

From the comparison principle for reflected processes follows \(\tilde{X}_s^{\tau, X_r} \leq \tilde{X}_s^{\tau, b(\tau)}\) for all \(s \geq \tau\). By the minimality property of reflected processes the original process \(X\) and the reflected process \(\tilde{X}^{\tau, X_r}\) coincide for all times before they hit the barrier, i.e. \(X_s = \tilde{X}_s^{\tau, X_r}\) for all \(s < \tau_b\). Moreover, the Markov property of the reflected process yields \(\tilde{X}_s^{\tau_b, b(\tau_b)} = \tilde{X}_s^{\tau, X_r}\) for \(s \geq \tau_b\).
The monotonicity of $z$ implies
\[
\mathbb{E} \left[ \int_0^\tau z(s, X_s) \, ds + \pi(\tau) \right] = \mathbb{E} \left[ \int_0^\tau z(s, X_s) \, ds + \int_\tau^T z(s, \tilde{X}_s^{\tau, b(\tau)}) \, ds \right] \\
\leq \mathbb{E} \left[ \int_0^\tau z(s, X_s) \, ds + \int_\tau^T z(s, \tilde{X}_s^{\tau, X_\tau}) \, ds \right] \\
= \mathbb{E} \left[ \int_0^\tau z(s, X_s) \, ds + \int_\tau^{\tau_b} z(s, \tilde{X}_s^{\tau, X_\tau}) \, ds + \int_{\tau_b}^T z(s, \tilde{X}_s^{\tau, b(\tau)}) \, ds \right] \\
= \mathbb{E} \left[ \int_0^\tau z(s, X_s) \, ds + \int_\tau^{\tau_b} z(s, X_s^{\tau, X_\tau}) \, ds + \int_{\tau_b}^T z(s, \tilde{X}_s^{\tau, b(\tau)}) \, ds \right] \\
= \mathbb{E} \left[ \int_0^{\tau_b} h(t, X_t) \, dt + g(\tau_b, X_{\tau_b}) + \pi(\tau_b) \right].
\]

3.3. All Implementable Stopping Rules are Cut-off Rules. For a fixed transfer $\pi$ and a fixed time-$t$ signal $x = X_t$ the agent’s continuation value $w^\pi(t, x)$ is the increase in expected payoff if the agent uses the optimal continuation strategy instead of stopping at time $t$
\[
w^\pi(t, x) = \sup_{\tau \in \mathcal{T}_t, T} \mathbb{E} \left[ \int_t^\tau h(s, X_s^{t,x}) \, ds + g(\tau, X_\tau^{t,x}) + \pi(\tau) \right] - g(t, x) - \pi(t).
\]
The continuation value satisfies the following properties.\footnote{The monotonicity of $x \mapsto w^\pi(t, x)$ and the resulting form of the continuation region is shown in \textcite{Jacka1992} for the case that $X$ is a Itô diffusion.}

\textbf{Lemma 6.} The mapping $x \mapsto w^\pi(t, x)$ is non-increasing and Lipschitz continuous.

By definition the continuation value is non-negative. If the continuation value is positive it is never optimal for the agent to stop. If the continuation value equals zero it is optimal to stop immediately but potentially there exists another strategy that stops later and is also optimal. Clearly if the agent is indifferent strategies that randomize over stopping and continuing are also optimal.

\textbf{Definition 7} (Minimal Stopping Rule). We call a stopping time $\tau$ which is implemented by the transfer $\pi$ minimal if and only if the agent stops the first time it is optimal to stop, i.e.
\[
\tau = \inf \{ t \geq 0 \mid w^\pi(t, X_t) = 0 \}.
\]
The restriction to minimal stopping rules allows us to excludes non-generic cases. The next proposition shows only cut-off rules can be implemented using a transfer $\pi$ if we assume that the agent stops the first time it is optimal for him.
**Proposition 8.** Let the stopping rule $\tau$ be implemented by the transfer $\pi$ and minimal then $\tau$ is a cut-off rule.

**Proof.** We fix a point in time $t \in [0, T]$ and introduce the stopping region $D_t = \{x \in \mathbb{R} | w^\pi(t, x) = 0\}$. Let $x \in D_t$ and $x' \geq x$. Then Lemma 6 implies that the continuation value $w^\pi(t, x')$ at $x'$ is smaller than the continuation value $w^\pi(t, x)$ at $x$ and hence $x'$ is in the stopping region $D_t$ as well. This implies that $D_t$ is an interval which is unbounded on the right. By Lemma 6 the function $x \rightarrow w^\pi(t, x)$ is continuous and hence $D_t$ is closed. Therefore we have $D_t = [b(t), \infty)$ for some $b(t) \in \mathbb{R}$. This implies that $\tau$ is a cut-off rule with barrier $b$

$$\tau = \inf\{t \geq 0 | X_t \in D_t\} = \inf\{t \geq 0 | X_t \geq b(t)\}.$$  

3.4. **A Principal-Optimal Policy is Implementable.** In this section we prove that a principal-optimal policy is implementable. A principal-optimal policy that maximizes the payoff of the principal

$$W(\tau) = \mathbb{E} \left[ \int_0^\tau \beta(t, X_t) dt + \alpha(\tau, X_\tau) \right].$$

Recall that we made no assumptions on the payoffs $a, b$ of the principal, apart from differentiability and the single-crossing assumption.

**Theorem 9.** There exists a principal-optimal stopping rule that is implementable.

**Proof.** Assume that the preferences of the principal and the agent are perfectly aligned $V \equiv W$. It follows from Proposition 8 that the minimal stopping rule implemented by the transfer of zero $\pi \equiv 0$ is a cut-off rule. As the preferences of the agent and the principal are the same it follows that the minimal principal-optimal stopping rule is a cut-off rule. By Theorem 5 all cut-off rules can be implemented and it follows that the minimal principal-optimal stopping rule is implementable.

3.5. **Uniqueness of The Payment.** In this section we derive conditions such that the payment implementing a cut-off rule $\tau_b$ is uniquely determined up to a constant.

**Proposition 10.** Let $X$ be a regular continuous diffusion ($\gamma = 0$ and $\sigma$ bounded away from zero) and the barrier $b$ be continuously differentiable, then there exists up to an additive constant at most one continuously differentiable transfer $\pi : [0, T] \rightarrow \mathbb{R}$ implementing a cut-off rule $\tau_b$.

**Proof.** Let $\pi$ denote a continuously differentiable transfer implementing the stopping rule $\tau_b$. We assume that $\pi(T) = 0$ and show that $\pi$ is uniquely determined. The value function is
defined by

\[ v^\pi(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E} \left[ \int_t^\tau h(s, X_s^{t,x}) ds + g(\tau, X_\tau^{t,x}) + \pi(\tau) \right] . \]

Since \( g, h \) and \( \pi \) are sufficiently smooth it follows that \( v^\pi \) is continuously differentiable in \( t \) and twice continuously differentiable in \( x \) for all \( x < b(t) \). Moreover, since \( \tau_b \) is an optimal stopping time in (3.3) the value function is also given by

\[ v^\pi(t, x) = \mathbb{E} \left[ \int_t^{\tau_b} h(s, X_s^{t,x}) ds + g(\tau_b, X_{\tau_b}^{t,x}) + \pi(\tau_b) \right] . \]

In particular, at every point \( x < b(t) \) below the barrier it is optimal to continue, which implies that

\[ \mathcal{L}v^\pi(t, x) + h(t, x) = 0. \]

On the barrier it is optimal to stop, which yields

\[ v^\pi(t, b(t)) = g(t, b(t)) + \pi(t) \]

for all \( t \in [0, T] \). Since the agent has to stop at time \( T \) we have for all \( x \in \mathbb{R} \)

\[ v^\pi(T, x) = g(T, x). \]

Moreover the smooth pasting principle implies that

\[ \partial_x v^\pi(t, b(t)) = \partial_x g(t, b(t)). \]

Observe that by Ladyzhenskaja et al. (1968, Theorem 5.3 in Chapter 4) there exists a unique solution to Equations (3.4), (3.6) and (3.7). As these three equations do not depend on the transfer \( \pi \) the value function does not depend on \( \pi \) neither. So for all continuously differentiable transfers \( \pi \) with \( \pi(T) = 0 \) the associated value functions coincide \( v^\pi = v \). But by Equation (3.5) \( \pi \) is uniquely determined by

\[ \pi(t) = v^\pi(t, b(t)) - g(t, b(t)). \]

4. Applications and Examples

4.1. Irreversible Investment and Regulation. In this section we present an application of our result to the classical irreversible investment literature. A pharmaceutical firm observes demand for a new product \( X \). The demand \( X_t \) is a geometric Brownian motion with drift

\[ dX_t = X_t \mu dt + X_t \sigma dW_t, \]

where \( 0 \leq \mu < r \) and \( \sigma > 0 \). If the firm starts selling the product at time \( t \) it pays continuous investment cost \( c : \mathbb{R}_+ \to \mathbb{R}_+ \) and receives the flow returns from selling the product. The
total expected payoff of the firm equals

\[ V(\tau) = \mathbb{E} \left[ -c(\tau) + \int_{\tau}^{\infty} e^{-rt} X_t \, dt \right] = \mathbb{E} \left[ -c(\tau) + e^{-r\tau} \frac{X_\tau}{r - \mu} \right]. \]

When timing its market entry decision the firm solves an irreversible investment problem. As famously shown in the irreversible investment literature (see for example Dixit and Pindyck (2008)) it is optimal for the firm to delay investment beyond the point where the net present value \( -c(t) + e^{-rt} \frac{X_t}{r - \mu} \) of investing becomes positive. While this behavior is optimal for the firm it is wasteful from a social perspective. Assume that the consumers discount at the same rate have an additional gain \( \Gamma > 0 \) from the pharmaceutical such that the overall benefit of society equals

\[ W(\tau) = V(\tau) + \mathbb{E} \left[ \int_{\tau}^{\infty} e^{-rt} \Gamma X_t \, dt \right] = \mathbb{E} \left[ -c(\tau) + e^{-r\tau} \frac{(1 + \Gamma)X_\tau}{r - \mu} \right]. \]

As the firm does not internalize the benefit of the consumers it will enter the market too late.

**Proposition 11.** Let \( \tau^V = \arg \sup_\tau V(\tau) \) be the optimal stopping time of the firm and \( \tau^W = \arg \sup_\tau W(\tau) \) be the socially optimal stopping time, then we have that \( \tau^W < \tau^V \).

A question that naturally arises is how a regulator could improve the incentives of the firm if demand is private information of the firm. By Theorem 1 we know that using a simple posted-price mechanisms the regulator can always implement the socially efficient market entry decision. In the next paragraph we will explicitly calculate the associated transfer \( \pi \).

Define \( g(t, x) = e^{-rt} \frac{x}{r - \mu} - c(t) \), the firm faces the maximization problem \( V(\tau) = \mathbb{E} [g(\tau, X_\tau)] \). The generator \( \mathcal{L}g \) is given by

\[ (\mathcal{L}g)(t, x) = -c'(t) - re^{-rt} \frac{x}{r - \mu} + e^{-rt} \frac{\mu x}{r - \mu} = -c'(t) - e^{-rt} x. \]

Clearly \( (\mathcal{L}g)(t, x) \) is decreasing in \( x \) and thus the dynamic single-crossing condition is satisfied. The transfer is given by

\[ \pi(t) = \mathbb{E} \left[ \int_t^T (\mathcal{L}g)(s, \tilde{X}_s^{t, b(t)}) \, ds \right] = \mathbb{E} \left[ \int_t^T c'(s) - e^{-rs} \tilde{X}_s^{t, b(t)} \, ds \right] = (c(T) - c(t)) - \phi_b(t, b(t)), \]
where \( \phi_b(t,x) = \mathbb{E} \left[ \int_t^T e^{-rs} \tilde{X}_s ds \right] \) is the expectation of the reflected process \( \tilde{X} \). The function \( \phi \) is the unique solution to the partial differential equation

\[
\begin{align*}
\phi_x(t,x) &= 0 \text{ for all } x = b(t) \\
\phi_t(t,x) + \phi_x(t,x) \mu_x + \phi_{xx}(t,x) \frac{x^2 \sigma^2}{2} &= e^{-rt} x \text{ for all } x < b(t).
\end{align*}
\]

This allows one to easily calculate \( \phi \) numerically.

### 4.2. A Revenue Maximization Example

This example presents a simple application of our results to revenue maximization. Let \( \tau : \mathbb{R}_+ \to \mathcal{T} \) the family of stopping times used in the revenue maximizing mechanism depending on the initial value. The setting is as in the previous section. It is a consequence of the envelope theorem that the agent’s value must be of the form

\[
V'(x_0) = \mathbb{E} \left[ e^{-r\tau(x_0)} \frac{X_\tau}{X_0} \right] = \mathbb{E} \left[ e^{-(r-\mu+\frac{\sigma^2}{2})\tau(X_0)+\sigma W_\tau} \right]
\]

By the revenue equivalence theorem the revenue of the principal equals

\[
\mathbb{E} \left[ e^{-r\tau} X_\tau \left( 1 - \frac{1 - F(X_0)}{f(X_0)X_0} \right) - c(\tau) \right].
\]

As the revenue of the principal depends on the initial value \( x_0 \) as well as on the final value \( X_\tau \) the principal wants to implement different stopping times for different initial values of the signal. The revenue maximal stopping time depending on the initial value \( X_0 \) is given by

\[
\tau^*(X_0) \in \arg \max_{\tau \in \mathcal{T}} \mathbb{E} \left[ e^{-r\tau} X_\tau \left( 1 - \frac{1 - F(X_0)}{f(X_0)X_0} \right) - c(\tau) \right].
\]

By Theorem 5 we can find a transfer \( \pi : \mathbb{R}_+ \times [0,T] \to \mathbb{R} \) depending on the initial value \( x_0 \) and the time the agent stopped such that given the agent reported \( x_0 \) at time zero he chooses the stopping time \( \tau^*(x_0) \) later. It remains to prove that the agent has incentives to report truthfully at time zero. To prove this we need show that the cross derivative of the agent’s value does not change its sign

\[
\frac{\partial}{\partial X_0} \mathbb{E} \left[ e^{-(r-\mu+\frac{\sigma^2}{2})\tau(X_0)+\sigma W_\tau} \right] \geq 0.
\]

A sufficient condition is that \( \tau(x) < \tau(x') \) for all \( x > x' \).

### 4.3. Constant Processes and the Relation to Static Mechanism Design

In this section we show that for the special case of constant processes our setting reduces to the classical static mechanism design setting analyzed among others by Mirrlees (1971) and Guesnerie and Laffont (1984). We will explicitly construct the reflected version \( \tilde{X} \) of the process \( X \). In this context the set of cut-off rules is equivalent to the set of monotone allocations and thus we recover the famous result that every monotone allocation is implementable.
Consider the case without flow transfers $h \equiv 0$. Let the signal $X$ be constant after time zero, i.e. $X_t = X_0$ for all $t \in [0, T]$. As the signal is constant after time zero the private information is one dimensional. Consequently the outcome of any (deterministic) stopping rule $\tau$ depends only on the initial signal $X_0$. Henceforth we define $\hat{\tau} : \mathbb{R} \to [0, T]$ as a function of the initial signal $\hat{\tau}(x) = \mathbb{E} [\tau | X_0 = x]$. As the outcome $\hat{\tau}(X_0)$ depends only on the initial signal $X_0$ we can define a transfer $\hat{\pi} : \mathbb{R} \to \mathbb{R}$ that only conditions on $X_0$ by $\hat{\pi}(x) = \pi(\hat{\tau}(x))$.

Implementability reduces to the following well known definition of static Implementability (Guesnerie and Laffont (1984), Definition 1)

**Definition 12** (Static Implementability). An allocation rule $\tau : \mathbb{R} \to [0, T]$ is implementable if there exists a transfer $\hat{\pi} : \mathbb{R} \to \mathbb{R}$ such that for all $x, x' \in X$

$$g(\hat{\tau}(x), x) + \hat{\pi}(x) \geq g(\hat{\tau}(x'), x) + \hat{\pi}(x').$$

As famously shown by Mirrlees (1971) and Guesnerie and Laffont (1984) any monotone decreasing allocation rule is implementable (Guesnerie and Laffont (1984), Theorem 2). Furthermore the transfers implementing $\hat{\tau}$ are of the from

$$\hat{\pi}(x) = \int_{x}^{X} \frac{\partial}{\partial x} g(\hat{\tau}(z), z) dz + g(\hat{\tau}(x), x) + c,$$

with some constant $c \in \mathbb{R}$.

In the following paragraph we show how our dynamic single-crossing condition, cut-off rules and the transfer simplify to the results well-known from static mechanism design. The generator $Lg$ equals

$$(Lg)(t, x) = \lim_{\Delta \searrow 0} \frac{1}{\Delta} \mathbb{E} [g(t+\Delta, X_{t+\Delta}) - g(t, x) | X_t = x]$$

$$= \lim_{\Delta \searrow 0} \frac{g(t+\Delta, x) - g(t, x)}{\Delta} = \frac{\partial}{\partial t} g(t, x).$$

Thus the dynamic single-crossing condition simplifies to $\frac{\partial}{\partial x} g(t, x)$ decreasing in $x$ for all $t \in [0, T]$ and all $x \in \mathbb{R}$. As the process $X$ is constant, for every barrier there exists a decreasing barrier that induces the same stopping time. Hence without loss of generality we restrict attention to decreasing barriers $b$. It is easy to check that for every decreasing barrier $b$ a version of the process $X$ reflected at $b$ is given by

$$\tilde{X}_{s}^{t, b} = \min\{x, b(t)\}.$$

From the fact that the reflected version of the process $\tilde{X}_{s}^{t, b(t)}$ started on the barrier at time $t$ stays on the barrier at all future times $s > t$ follows that it is deterministic and given by

$\tilde{X}_{s}^{t} = \min\{x, b(t)\}$.\footnote{Both Mirrlees and Guesnerie & Laffont analyze a more general setting as they do not assume that the agent is risk-neutral or put differently that his utility is linear in the transfer.}
\( \tilde{X}_{s}^{t,b(t)} = b(s) \). Hence the transfer implementing the stopping time \( \tau_b \) equals

\[
\pi(t) = \mathbb{E} \left[ \int_{t}^{T} (\mathcal{L}g)(s, \tilde{X}_{s}^{t,b(t)}) ds \right] = \mathbb{E} \left[ \int_{t}^{T} \frac{\partial}{\partial t} g(s, b(s)) ds \right] = \int_{t}^{T} \frac{\partial}{\partial t} g(s, b(s)) ds .
\]

Finally we show that the transfer \( \pi(\tau(x)) \) equals the transfer \( \hat{\pi}(x) \) up to a constant.

**Proposition 13.** Let \( \tau : [\underline{x}, \overline{x}] \rightarrow [0, T] \) be measurable and non-increasing then \( \hat{\pi}(x) = \pi(\tau(x)) \) for all \( x \in [\underline{x}, \overline{x}] \).

**Proof.** Recall that \( \hat{\pi} \) is given by

\[
\hat{\pi}(x) = \int_{\underline{x}}^{x} \partial_x g(\hat{\tau}(z), z) dz - g(\hat{\tau}(x), x) + c,
\]

for some constant \( c \in \mathbb{R} \). We fix \( x \in [\underline{x}, \overline{x}] \) and compute

\[
\hat{\pi}(x) - c = \int_{\underline{x}}^{x} \partial_x g(\hat{\tau}(z), z) - \partial_x g(\hat{\tau}(x), z) dz - g(\hat{\tau}(x), x)
\]

Then Fubini’s Theorem implies

\[
\hat{\pi}(x) - c = \int_{\tau(x)}^{\tau(x)} \int_{\underline{x}}^{b(s)} \partial_x \partial_t g(s, z) dz ds - g(\tau(x), x)
\]

\[
= \int_{\tau(x)}^{\tau(x)} \partial_t g(s, b(s)) - \partial_t g(s, x) ds - g(\tau(x), x)
\]

\[
= \int_{\tau(x)}^{\tau(x)} \partial_t g(s, b(s)) ds - g(\tau(x), x).
\]

Choosing \( c = \int_{\tau(x)}^{\tau(x)} \partial_t g(s, b(s)) ds + g(\tau(x), x) \) yields \( \hat{\pi}(x) = \pi(\tau(x)) \). \( \Box \)

### 4.4. Monotone Increasing Processes and Decreasing Cut-Off

In this section we restrict attention to increasing processes and stopping times with decreasing cut-off \( b \). It is easy to check that for every decreasing barrier \( b \) a version of the process \( X \) reflected at \( b \) is given by

\[
\tilde{X}_{s}^{t,x} = \min\{x, b(t)\}.
\]

From the fact that the reflected version of the process \( \tilde{X}_{s}^{t,b(t)} \) started on the barrier at time \( t \) stays on the barrier at all future times \( s > t \) follows that it is deterministic and given by \( \tilde{X}_{s}^{t,b(t)} = b(s) \). Hence the transfer implementing the stopping time \( \tau_b \) equals

\[
(4.1) \pi(t) = \mathbb{E} \left[ \int_{t}^{T} (\mathcal{L}g)(s, \tilde{X}_{s}^{t,b(t)}) ds \right] = \mathbb{E} \left[ \int_{t}^{T} (\mathcal{L}g)(s, b(s)) ds \right] = \int_{t}^{T} (\mathcal{L}g)(s, b(s)) ds .
\]
4.5. Search with Recall. In the next section we apply our result to (job) search with recall. Offers arrive according to a Poisson process $(N_t)_{t \in [0,T]}$ with constant rate $\lambda > 0$. Offers $(w_i)_{i \in N}$ are identically and independently distributed according to the distribution $G : \mathbb{R} \to \mathbb{R}$. $X_t^{s,t}$ is the best offer the agent got in the past, where we assume that the agent started with an offer of $x \in \mathbb{R}$ at time $s$:

$$X_t^{s,x} = \max\{w_i | i \leq N_t - 1\} \lor z. \footnote{18}$$

We assume that the agent receives a final payoff $g(t,x) = e^{-rt}x$ and a flow payoff of zero. The generator of $g$ equals

$$(Lg)(t,x) = -rx + \lambda \int_{x}^{\infty} (1 - G(z))dz. \footnote{18}$$

As $G(z) \leq 1$ the generator is monotone decreasing in $x$ and thus the single-crossing condition is satisfied. By definition the process $X$ is monotone increasing and by equation (4.1) the transfer that implements a monotone cut-off $b$ is given by

$$\pi(t) = \int_t^T (Lg)(s,b(s))ds = \int_t^T -rb(s) + \lambda \int_{b(s)}^{\infty} (1 - G(z))dz \, ds. \footnote{18}$$

4.6. Application to Optimal Stopping. As an immediate consequence of Theorem 5 we establish a link between reflected processes and optimal stopping rules. If a stopping rule $\tau^*$ is implemented by a constant transfer, then $\tau^*$ is optimal in the stopping problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[ \int_0^\tau h(t,X_t)dt + g(\tau,X_{\tau}) \right] \footnote{18}$$

This observation leads to the following purely probabilistic sufficient condition for optimal stopping rules.

**Corollary 14.** Let $b : [0,T] \to \mathbb{R}$ be a barrier such that the reflected process satisfies

$$\mathbb{E} \left[ \int_t^T h(s,\tilde{X}_t^{s,b(t)} + (Lg)(s,\tilde{X}_t^{s,b(t)})ds \right] = 0 \footnote{18}$$

for all $t \in [0,T]$. Then the associated cut-off rule $\tau_b$ solves the optimal stopping problem (4.2).

**Conclusion**

We have shown under weak assumptions that in optimal stopping settings a stopping rule that is optimal for the principal can be implemented by a simple posted-price mechanism. The new approach we introduced is a forward construction of transfers. The transfer is the expected payoff evaluated at the process reflected at the time-dependent cut-off one wants to implement.
It is an interesting question for future research whether a similar approach can be used in environments where non cut-off rules are of interest. Two such situations arise naturally.

First, with multiple agents, the socially-efficient cut-off depends not only on time but also on the signals of the other agents. We think that a generalization of the method developed in this paper can be used to implement the principal-optimal behavior in this multi-agent settings. As the cut-off depends on the other agents’ signals, one needs to define processes reflected at stochastic barriers. If one defines payments as expected payoffs evaluated at the process reflected at the stochastic cut-off, we conjecture that our implementability result (Theorem 5) holds also for those generalized stochastic cut-offs. The payment, however, will depend on the other agents’ signals and, thus, the mechanism will require constant communication between the principal and the agents.

A second, more challenging, problem is revenue maximization. For special cases the revenue maximizing policy depends only on the initial and the current signal. In those cases one might be able to use the method developed here for revenue maximization by offering at time zero a menu of posted-price mechanisms.

Those extensions illustrate that posted-price transfers without communication are often insufficient in more general dynamic setups. We hope, however, that due to its simplicity and the explicit characterization of transfers, our result will prove useful to introduce private information into the wide range of economic situations that are modeled as single-agent optimal stopping problems.

**APPENDIX**

**Appendix A.** We impose the following assumptions on the coefficients in (2.1). We assume that $\mu, \sigma$ and $\gamma$ are Lipschitz continuous, i.e. there exist positive constants $L$ and $L(z)$ such that

\[
|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|
\]

\[
|\gamma(t, x, z) - \gamma(t, y, z)| \leq L(z)|x - y|
\]

for all $t \in [0, T]$ and $x, y \in \mathbb{R}$. Moreover, we assume that these functions are of linear growth, i.e. there exist positive constants $K$ and $K(z)$ such that

\[
|\mu(t, x)| + |\sigma(t, x)| \leq K(1 + |x|)
\]

\[
|\gamma(t, x, z)| \leq K(z)(1 + |x|)
\]

for all $t \in [0, T]$ and $x \in \mathbb{R}$. Furthermore the constants $L(z)$ and $K(z)$ satisfy

\[
\int_{\mathbb{R}} K(z)^p + L(z)^p \nu(dz) < \infty
\]
for all \( p \geq 2 \).

**Proof of Lemma 6.** Let us first show monotonicity of \( x \mapsto w^\pi(t, x) \). By Dynkin’s formula the continuation value satisfies

\[
    w^\pi(t, x) = \sup_{\tau \in T \cap [t,T]} \mathbb{E} \left[ \int_t^\tau h(s, X^t_{s}, x) + \mathcal{L}g(s, X^t_{s}, x)ds + \pi(\tau) - \pi(t) \right]
\]

for all \((t, x) \in [0, T] \times \mathbb{R}\). The dynamic single-crossing condition and Equation (2.2) imply that the flow payoffs given that \( X \) started in a higher level \( x' \geq x \) are smaller than the flow payoffs if it started in \( x \)

\[
    h(s, X^t_{s}, x') + \mathcal{L}g(s, X^t_{s}, x') \leq h(s, X^t_{s}, x) + \mathcal{L}g(s, X^t_{s}, x)
\]

for all \( s \geq t \). Hence, we have \( w^\pi(t, x') \leq w^\pi(t, x) \). For the Lipschitz continuity fix \( t \in [0, T] \) and \( x, y \in \mathbb{R} \). We have

\[
    |w^\pi(t, x) - w^\pi(t, y)| \leq \sup_{\tau \in T \cap [t,T]} \mathbb{E} \left[ \int_t^\tau |h(s, X^t_{s}, x) - h(s, X^t_{s}, y)| ds + |g(s, X^t_{s}, x) - g(s, X^t_{s}, y)| + |g(t, x) - g(t, y)| \right].
\]

Let \( C \) denote the Lipschitz constants of \( h \) and \( g \), i.e.

\[
    |h(t, x) - h(t, y)| + |g(t, x) - g(t, y)| \leq C|x - y|
\]

Then we have

\[
    |w^\pi(t, x) - w^\pi(t, y)| \leq 2C\mathbb{E} \left[ \sup_{s \in [t,T]} |X^t_{s} - X^t_{s}| + |x - y| \right].
\]

By Theorem 3.2 in Kunita there exists a constant \( \tilde{C} \) such that \( \mathbb{E} \left[ \sup_{s \in [t,T]} |X^t_{s} - X^t_{s}| \right] \leq \tilde{C}|x - y| \), which yields the claim. \( \square \)

**Appendix B.**

**Proof of Proposition 4.** For case b) it is straightforward to verify that \( \tilde{X}^\tau_{s, \eta} = \min\{X^\tau_{s, \eta}, b(s)\} \) satisfies all the conditions from the definition of a reflected process. For case a) we show that the solution to a reflected stochastic differential equation (RSDE) is a reflected version of \( X \). A solution to a RSDE with initial condition \((t, \eta) \) is a pair of processes \((\tilde{X}, l) \) which satisfies the equation

\[
    \tilde{X}_s = \eta + \int_t^s \mu(r, \tilde{X}_r)dr + \int_t^s \sigma(r, \tilde{X}_r)dW_r - l_s,
\]

Moreover, \( \tilde{X} \) is constrained to stay below the barrier \( \tilde{X}_s \leq b(s) \) and the process \( l \) is non-decreasing and minimal in the sense that it only increases on the barrier, i.e. \( \int_t^T b(s) - \tilde{X}_sdl_s = 0 \). Existence of \((\tilde{X}, l) \) follows from Slominski and Wojciechowski (2010) Theorem.
3.4). Conditions (3.1), (3.1) and (3.1) are satisfied by definition. Furthermore pathwise uniqueness holds, which implies condition (3.1). Note that the solution to the unreflected SDE (2.1) solves the RSDE for \( s < \tau \). As the solution to the RSDE is unique (3.1) follows. It remains to verify conditions (3.1) and (3.1). For \( \xi_1, \xi_2 \in \mathbb{R} \) we write \((\tilde{X}^i, l^i) = (X_0, \xi_i, l^{0,i})\), \((i = 1, 2)\) and introduce the process \(\Delta_s = \tilde{X}_s^1 - \tilde{X}_s^2\). Applying the Meyer-Itô formula [Protter 2005, Theorem 71, Chapter 4] to the function \( x \mapsto \max(0, x)^2 \) yields

\[
\max(0, \Delta_s)^2 = \max(0, \Delta_0)^2 + 2 \int_0^s 1_{\{\Delta_t > 0\}} \Delta_t d\Delta_t^c \\
+ \int_0^s 1_{\{\Delta_t > 0\}} d\langle \Delta \rangle_t^c + \sum_{0 < r \leq s} \max(0, \Delta_r)^2 - \max(0, \Delta_{r-})^2,
\]

where \( \Delta^c \) denotes the continuous part of \( \Delta \). For the last integral the Lipschitz continuity of \( \sigma \) implies

\[
\int_0^s 1_{\{\Delta_t > 0\}} d\langle \Delta \rangle_t^c = \int_0^s 1_{\{\Delta_t > 0\}} (\sigma(r, \tilde{X}_r^1) - \sigma(r, \tilde{X}_r^2))^2 dr \leq K \int_0^s \max(0, \Delta_r)^2 dr.
\]

The first integral decomposes into the following terms, which we will consider successively. By the Lipschitz continuity of \( b \) we have

\[
2 \int_0^s 1_{\{\Delta_t > 0\}} \Delta_t (b(r, \tilde{X}_r^1) - b(r, \tilde{X}_r^2)) dr \leq 2K \int_0^s \max(0, \Delta_r)^2 dr.
\]

Moreover, it follows from the Lipschitz continuity of \( \sigma \) that the process

\[
s \mapsto 2 \int_0^s 1_{\{\Delta_t > 0\}} \Delta_t (\sigma(r, \tilde{X}_r^1) - \sigma(r, \tilde{X}_r^2)) dW_r
\]

is a martingale starting in 0. Hence, it vanishes in expectation. Next, we have

\[
-2 \int_0^s 1_{\{\Delta_t > 0\}} \Delta_t dl_t^1 \leq 0
\]

and

\[
2 \int_0^s 1_{\{\Delta_t > 0\}} \Delta_t dl_t^2 \leq 2 \int_0^s 1_{\{\tilde{X}_t^1 > b(r)\}} \Delta_t dl_t^2 = 0.
\]

Putting everything together, we obtain

\[
\mathbb{E}[2 \int_0^s 1_{\{\Delta_t > 0\}} \Delta_t d\Delta_t^c] \leq 2K \int_0^s \mathbb{E}[\max(0, \Delta_r)^2] dr.
\]

Considering the jump terms, assume that there exists \( r \in (0, s] \) such that \( \max(0, \Delta_r)^2 > \max(0, \Delta_{r-})^2 \). This implies \( \Delta_r > 0 \) and \( \Delta_r > \Delta_{r-} \). Since \( \tilde{X}_i \) jumps if and only if \( l^i \) jumps \((i = 1, 2)\), this is equivalent to \( \tilde{X}_1 > \tilde{X}_2 \) and \( l_r^2 - l_r^2 > l_r^1 - l_r^1 \). We have \( l_r^2 - l_r^2 > 0 \), because \( l ^1 \) is non-decreasing. Hence, \( l^2 \) jumps at \( r \), which implies that \( \tilde{X}_r^2 = b(r) \). Thus, we
obtain the contradiction \( \tilde{X}_r^1 > b(r) \). This implies
\[
\mathbb{E}[\max(0, \Delta_s)^2] \leq \max(0, \Delta_0)^2 + 3K \int_0^s \mathbb{E}[\max(0, \Delta_r)^2] dr.
\]
Then Gronwall's Lemma yields
\[
\mathbb{E}[\max(0, \Delta_s)^2] \leq C \max(0, \Delta_0)^2 \text{ for some constant } C > 0.
\]
For \( t > 0 \) performing the same computations with the expectation taken conditional to \( \mathcal{F}_t \) yields for \( \xi_1, \xi_2 \in L^2(\mathcal{F}_t) \)
\[
\mathbb{E}[\max(0, \tilde{X}_s^1 - \tilde{X}_s^2)^2 | \mathcal{F}_t] \leq C \max(0, \xi_1 - \xi_2)^2
\]
for all \( s \in [t, T] \). If \( \xi_1 \leq \xi_2 \) this directly yields (3.1). Claim (3.1) follows by the same argument.
References


