RISK-TAKING IN CONTESTS
THE IMPACT OF FUND-MANAGER COMPENSATION ON INVESTOR WELFARE

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Abstract. We analyze welfare consequences of competition between agents who get paid based on their relative performance. Each agent chooses a distribution from a convex set and wins a prize which only depends on her rank relative to her peers. We allow for completely general prize structures and find that the equilibrium distributions induce risk in the outcome. This risk increases in the sense of second order stochastic dominance when competition between the agents increases.

A leading example of such a situation is the competition between fund-managers whose compensation depends on how well they performed relative to other fund-managers in the same investment category. Our model is general enough such that it captures dynamic financial markets models à la Black Scholes. In the unique symmetric equilibrium, fund-managers use randomized investment strategies which endogenously create risk. This risk is unrelated to the uncertainty generated by the assets traded in the underlying financial market. We find that this risk is excessive as there exists a trading strategy that dominates the equilibrium return distribution in the sense of second order stochastic dominance.

This excessive risk taking of fund-managers leads to welfare losses if investors are risk-averse. Numerical examples indicate that welfare losses are substantial. Finally, we show that an increase in competition between fund-managers increases welfare losses.

Keywords: Dynamic Contests, Risk-Taking Behavior, All-Pay Contest, Managed Funds, Endogenous Risk, Black-Scholes

JEL: G11, D81, G23

1. INTRODUCTION

This paper analyzes the risk-taking behavior of symmetric agents who compete in a general rank-order contest. Each agent chooses a distribution over scores conditional on an external state from a convex set. Agents get ranked according to their score and each agent receives a prize which is only a function of her score. We characterize the unique symmetric equilibrium and find that the riskiness of outcomes increases in the sense of second order stochastic dominance when competition between the agents increases.

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While there are many situations where the riskiness of contest outcomes potentially plays an important role one which is of particular importance from a social welfare perspective is the competition between fund-managers.

**Application to Managed Funds.** Over the past decade, American investors have increasingly turned to managed funds to save for retirement. In 2013, 46.3% of US households held investments into managed funds valued at 17.1 trillion US dollar.¹ Shareholders of a fund delegate all its investments decisions to the manager. Given managed funds play a central role for the allocation of retirement money, it is important to understand how fund-managers make investment decisions. Are the incentives that fund-managers face in line with their investors’ interests? Our setup allows us to explicitly model the incentives fund-managers face when competing for investors’ money and characterizes in which assets a fund-manager invests to maximize her compensation.

A fund manager’s compensation is mostly determined as a fixed fraction of assets under management (cf Brown, Harlow, and Starks [1996]). In order to maximize her compensation a fund-manager aims to maximize the inflow of money into her fund. Empirically, funds which perform best experience the highest inflow of money (cf Sirri and Tufano [1992, 1998]). This can be explained by investors’ reliance on rankings like the Morningstar or NY Times rating which are based on past relative performance²⁻³ (Reuter and Zitzewitz [2010], Guercio and Tkac [2008]).⁴ As fund-manager’s compensation is linked to relative performance they should behave as if they are competing in a tournament. The empirical literature extensively documents that fund-managers react to those incentives. For example fund-managers increase the riskiness of their portfolio in situations where doing so increases their expected inflow of money and compensation (Chevalier and Ellison [1997], Kempf and Ruenzi [2008], Brown, Harlow, and Starks [1996]).

We characterize the investment strategies fund-managers pursue in equilibrium. The model predicts that when fund-managers compete for new investors, they choose investment strategies which are inefficiently risky. More precisely, we find that in equilibrium fund-managers use mixed strategies which create endogenous risk, i.e. risk entirely unrelated to the risk of the underlying financial markets. This randomization leads to a distribution of fund returns which is second order

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²The SEC on managed-funds: “Advertisements, rankings, and ratings often emphasize how well a fund has performed in the past. But studies show that the future is often different. This year’s “number one” fund can easily become next year’s below average fund” (see http://www.sec.gov/investor/pubs/sec-guide-to-mutual-funds.pdf).
³Morningstar rates mutual funds from one to five stars based on how well they’ve performed (after adjusting for risk and accounting for all sales charges) in comparison to similar funds. Within each Morningstar Category, the top 10% of funds receive five stars, the next 22.5% four stars, the middle 35% three stars, the next 22.5% two stars, and the bottom 10% receive one star.” see http://screen.morningstar.com/FundInsights Definitions/RiskRating2.html
⁴Even though the past performance of a fund has little predictive power for future performance (see for example Carhart [1997]), Berk and Green [2004] show that return chasing behavior can be rational in a situation where fund-manager have different abilities.
stochastically dominated by the return distribution from using the long-run growth maximizing strategy which in the Black-Scholes model takes the form of a simple index strategy. In the Black-Scholes model risk-averse investors prefer the return distribution which is generated by investing a fixed fraction of her wealth in the risky asset to the return distribution fund managers’ symmetric equilibrium strategy induces. We show that fund-managers create more risk when the market returns are high. Furthermore, the amount of endogenous risk created in equilibrium increases as competition between the fund-managers increases.

To quantify the amount of endogenous risk created in equilibrium we compare the standard deviation of the equilibrium return distribution to the standard deviation of the long-run growth maximizing strategy and find that equilibrium returns have a significantly higher standard deviation. The bound we derive is robust in the sense that it is independent of the market structure.

Related Literature. This paper generalizes the model of competition introduced in Seel and Strack [2013b] which was subsequently extended in Seel and Strack [2013a], Feng and Hobson [2014, 2015]. In this stream of literature each contestant choses a stopping time and the contestant with the highest outcome wins. The constraint to stopping times is equivalent to a constraint on the first moment of the outcome distribution and thus similar to the space of restrictions considered here. We extend this literature in two dimensions: first we allow for a completely general prize structure. Second we allow the distribution over outcomes to depend on a random state common to all players. The later extension allows us to incorporate standard models of financial markets which could not be incorporated in earlier models. The first extension is important as it allows to do comparative statics of the risk-taking behavior with respect to the compensation structure.

As fund-managers take investment decisions at every point in time our model relates to the literature on dynamic contests without interim feedback on the opponents actions and outcomes Taylor [1995]. Similar to Bimpikis, Ehsani, and Mostagir [2014], Halac, Kartik, and Liu [2014] we allow the set of feasible outcome distributions to depend on an external state common to the agents. While in Bimpikis et al. and Halac et al. the external state describes the feasibility of the project, it describes the evolution of stock market prices in our application to managed funds.

As we show in Section 7 that symmetric equilibrium distributions of our model correspond to symmetric equilibria of the all-pay auction the literature on full-information all pay auctions is closely related Barut and Kovenock [1998], Hillman and Samet [1987], Siegel [2009, 2010]. For the symmetric case our results extend the literature doing comparative statics in the compensation structure of an all-pay contest Barut and Kovenock [1998], Bulow and Levin [2006], Siegel [2010], González-Díaz and Siegel [2013], Xiao [2015] from linear, quadratic and homogeneous prize structures to arbitrary prize structures and finds that competition increases the dispersion of bids.
Closely related to the managed-funds application Basak, Pavlova, and Shapiro [2007] study risk-taking incentives of fund-managers in a Black-Scholes model when fund managers are evaluated relative to a benchmark index similar to the Kelly strategy considered in Section 6. Different from our analysis they do not model relative performance concerns, but allow for the incentive to beat a benchmark which we do not model. Basak and Makarov [2014] analyze the competition between two (potentially asymmetric) fund-managers in a Black Scholes model where the compensation depends on the ratio of the two fund-managers return to the power of a constant. An important difference is that returns of the other fund-manager are observed at an intermediate time. They provide transparent sufficient conditions on their compensation structure such that the game played between fund-managers admits a (unique) pure strategy equilibrium and characterize it. As this equilibrium is in pure strategies fund-managers do not create endogenous risk in equilibrium as they do in our model where compensation is based purely on fund-managers rank. They farsightedly conclude that “behavior in the mixed equilibrium is notably different from that in the pure equilibrium, implying that the issue of whether a pure equilibrium exists is relevant not only from a theoretical perspective but also from a practical perspective” and analyze mixed strategy equilibria in a one period setting where the stock either goes up by a fixed percentage. They state that “it does not appear to be possible to characterize a mixed equilibrium analytically in our continuous-time framework given that the strategy space is infinite-dimensional, and even numerically this task seems daunting. To circumvent this difficulty, we turn to a simpler binomial setting”.

The present study extends their results on mixed-strategy equilibria to the continuous-time setting for general compensation structures that are a function only of the rank under the restriction to symmetric fund-managers and no intermediate observability.\(^5\)

The paper proceeds as follows: Section 2 introduces the general model of contest when agents choose distributions. Section 2.1 and 2.3 argue that competition between fund-managers in standard models of complete financial markets (like the Black-Scholes model) is a special case of the general model if fund-managers can introduce idiosyncratic risk in their portfolio outcomes. Section 3 derives the unique symmetric equilibrium of the game. Section 4 analyzes investors’ equilibrium welfare and finds that the endogenous risk introduced in equilibrium substantially reduces investors welfare. In Section 6 we present a Black-Scholes example. Section 5 does comparative statics in the compensation structure and finds that an increase in competition between fund-managers leads to an increase in welfare losses for the investors. Section 7 discusses the relation to all-pay contests. Section 8 concludes. Most proofs are relegated to the Appendix.\(^5\)

\(^5\)Basak and Makarov [2014] propose this extension in their conclusion: “another natural extension of our framework would be to incorporate flow-performance relations where money flows depend on discrete rankings, leading to discontinuities in the managers’ objective functions”.
2. Model

We first describe an abstract model and then prove in section 2.1 and 2.3 that the competition between fund-managers in many standard models of dynamic financial markets especially the Black-Scholes model is a special case of the general model. There are \( n \geq 2 \) agents. There is a state \( S \in S \) - the outcome of the financial market - which is distributed according to \( \rho \in \Delta(S) \). Furthermore, there exists a measure \( \rho^* \in \Delta(S) \) which is mutually absolutely continuous to \( \rho \). Each agent \( i \in \{1, \ldots, n\} = N \) chooses a conditional distribution over outcomes \( X^i \in \mathbb{R}_+ \) from the set

\[
Q \triangleq \left\{ F : \mathbb{R}_+ \times S \to [0, 1] : \int_S \mathbb{E}^F [X^i \mid S = s] \, d\rho^*(s) \leq 1 \right\}.
\]

Agents are ranked according to their performance and ties are broken with equal probability.\(^6\) Denote by \( R^i \in N \) the rank of agent \( i \), i.e. \( R(i) \geq R(j) \iff X^i \geq X^j \). Each agent is risk-neutral and receives a payoff of \( b_{R(i)} \in \mathbb{R} \). The vector of prizes \( b \in \mathbb{R}^n \) is ordered \( b_1 < b_2 < \ldots < b_n \) and at least one inequality is strict, i.e. \( b_1 < b_n \).\(^7\) An equilibrium of the game is a vector of distributions \((F^1, \ldots, F^n)\) such that for all \( i \in N \) and all \( \tilde{F} \in Q \)

\[
\mathbb{E}^{(F^i, F^{-i})}[b_{R(i)}] \geq \mathbb{E}^{(F, F^{-i})}[b_{R(i)}].
\]

To motivate the study of this game we explain in the next section how it is equivalent to the competition of fund-managers in models of complete financial markets. Theorem 2 derives the unique symmetric equilibrium of this game.

2.1. Competition between Fund-Managers and Compensation. From now on we call agents fund-managers. Fund-managers are evaluated based on the return their trading activities up to time \( T \) generated. We assume without loss of generality that each fund-manager is initially (at time zero) endowed with one unit of assets under management. Each fund-manager \( i \) chooses a trading strategy \( \beta^i \) which induces a random final value of her portfolio \( X^i \). A vector of trading strategies \( \beta = (\beta^1, \beta^2, \ldots, \beta^n) \) induces a probability measure over the vector of final portfolio values \( (X^1, X^2, \ldots, X^n) = X \in \mathbb{R}^n \). As we normalized the initial assets of each fund-manager to one \( X^i - 1 \) equals the return of fund-manager \( i \). For example \( X^i = 1.05 \) means that fund-manager \( i \) earned a return of five percent on the assets under her management.

Denote by \( \mathbb{P}^\beta [\cdot] \) the probability measure induced by the vector of strategies \( \beta \) and by \( \mathbb{E}^\beta [\cdot] \) the corresponding expectation operator. Let \( R(i) \in N \) be the rank of fund-manager \( i \) according to the return she generated, i.e.

\[
X^{R^{-1}(1)} \leq X^{R^{-1}(2)} \leq \ldots \leq X^{R^{-1}(n)}.
\]

\(^6\)As shown in Theorem 2 the probability of a tie equals zero in the symmetric equilibrium.

\(^7\)The dynamic contest models in the previous literature \([Feng and Hobson, 2014, 2015, Seel and Strack, 2013a,b]\) analyze the special case where the state is a singleton \( S = \{s\} \) and only the best performing agent receives a prize \( b = (0, \ldots, 0, 1) \).
$R^{-1}(1) \in N$ is the fund-manager who generated the highest return, $R^{-1}(2)$ is the fund-manager who generated the second highest return and so on. Each fund-manager $i$ receives a compensation
\[ b_{R(i)} \in \mathbb{R} \]
which depends only on her rank $R(i)$. If multiple fund-managers generated the same return the ties are broken by assigning ranks randomly with equal probability to the fund-managers who performed equally well. We assume the vector $b \in \mathbb{R}^n$ to be ordered
\[ b_1 \leq b_2 \leq \ldots \leq b_n \]
such that fund-managers who performed better receive a higher compensation. To ensure that there is competition we assume that at least one inequality is strict, i.e. $b_1 < b_n$. For future reference denote by $\overline{b} = \frac{1}{n} \sum_{i=1}^{n} b_k$ the average compensation.

In a firm internal competition the reward from outperforming your peers might be explicit in the form of a promotion to a higher rank or a bonus. In a situation where fund-managers compete for the inflow of investors money the prize might be implicit and given by the higher future compensation resulting from the larger market share the fund captures.

To model the fund-managers behavior when competing for prizes we assume that they behave rationally and maximize their expected compensation taking into account the behavior of their competitors. As fund-managers are identical we focus on symmetric Nash equilibria.

**Definition 1** (Financial Market Equilibrium). A symmetric Nash equilibrium is a vector of trading strategies $\beta \in \mathcal{A}^n$ such that each fund-manager $i \in N$ maximizes her expected compensation,
\[ \mathbb{E}^{(\beta,\beta^{-}')}[b_{R(i)}] = \sup_{\beta} \mathbb{E}^{(\beta,\beta^{-}')}[b_{R(i)}]. \]

As an illustration consider the compensation vector $b = (0, \ldots, 0, 1)$ which awards a prize only to the best performing fund-manager. For this compensation structure each fund-manager $i$ maximizes the probability that she outperforms all her peers $\mathbb{P}[R(i) = n]$.

2.2. The Financial Market. Up to now we left the structure of the financial market implicit in the definition of trading strategies. This section specifies a general model of financial markets. There is a finite number $m \geq 1$ of financial assets. We denote by $S \in \mathcal{S}$ the state of the market at time $T \in \mathbb{R}_+$. For many financial market models (especially the Black Scholes model) $S \subseteq \mathbb{R}_+^m$ and $(S_1^T, \ldots, S_m^T)$ is simply the vector of market prices of the assets at time $T$. We denote by $\rho \in \Delta(S)$ the finite physical probability measure which describes the probability that the state of the market

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8This allows for risk-aversion of fund-managers: If fund-managers are not risk-neutral $b_r$ equals the expected utility from prize $r \in N$.

9In general one can always define the state to be the space of all paths over asset prices on the time interval $[0, T]$. The reason that it suffices to consider final prices $S_T$ in the Black-Scholes model is that the probability measure induced over paths induced by the geometric Brownian motion depends only on the final values.
at time $T$, i.e. for every set $M \subseteq S$

$$\mathbb{P}[S \in M] = \int_M d\rho(s).$$

Intuitively, $\rho$ is the probability measure over price movements in the underlying financial market. As we left $\rho$ general assets might be correlated in arbitrary ways.

We assume that there is a finite measure $\rho^* \in \Delta(S)$ called the pricing measure such that the initial wealth $\Pi(X) > 0$ necessary to construct a portfolio which pays out the (random) payoff $X(s)$ if the market is in state $s$ at time $T$ is given by its expected value under the pricing measure

$$(1) \quad \Pi(X) \triangleq \int_S \mathbb{E}[X(s)] d\rho^*(s).$$

We assume that there exists a function $\Gamma : S \to (0, \infty)$ such that $\Gamma$ denotes the relative likelihood of the state $s$ under the pricing measure $\rho^*$ relative to the physical measure $\rho$, i.e. for all $M \subseteq S$

$$\int_M d\rho(s) = \int_M \Gamma(s) d\rho^*(s).$$

Throughout we assume that the market is complete in the sense that any random positive payoff profile can be replicated:

**Assumption 1 (Randomly Complete Market).** For every random payoff $X : S \to \Delta(\mathbb{R}_+)$ such that the price $\Pi(X)$ is finite there exists a trading strategy that generates $X$.

For deterministic payoff profiles $X : S \to \mathbb{R}_+$ Assumption 1 is the standard assumption of complete markets. Market completeness is satisfied in many standard models of financial markets especially the Black and Scholes [1973] model in continuous time, or models where asset prices follow a Binomial random walk. In the context of those models market completeness implies that there are no-short selling constraints or transaction cost.

Randomly complete markets is a stronger assumption than complete markets as not only deterministic, but also random payoff profiles can be replicated by (potentially randomized) trading strategies. Essentially, random completeness ensures that the space of investment/trading strategies is sufficiently rich such that fund-managers can introduce non market related randomness in their portfolio outcome if they want to. As we show in Section 2.3 the standard assumption of complete markets plus allowing for mixed strategies is enough to ensure random completeness in the Black-Scholes model.\footnote{Mathematically, this assumption is equivalent that $\rho$ is absolutely continuous with respect to $\rho^*$ and $\Gamma$ is called the Radon-Nykodym derivative.}

We make the assumption of randomly complete markets as it allows us to analyze the competition between fund-managers without any further assumption on the market structure: The market

\footnote{Note, that not every complete financial market model is also randomly complete. For example an model with two states $S = \{1, 2\}$ (values of a risky asset) and Arrow Debreu securities is complete, but not randomly complete as the payoff in one state completely determines the payoff in the other state.}
model could be dynamic or static, set up in continuous time or discrete time, have arbitrarily many assets which are correlated in arbitrary ways. In that sense our results are robust to the fine details of the underlying financial market.

2.3. **The Black-Scholes Model is Randomly Complete.** To illustrate the market setup we now show that the classical continuous time Black-Scholes model with a risky and a safe asset is a special case of our general market model.\(^{12}\) There is a single risky asset, whose value \((S_t)_{t \in [0,T]}\) evolves according to a geometric Brownian motion with drift \(\mu \in \mathbb{R}_+\) and variance \(\sigma \in \mathbb{R}_+\) on \([0,T]\)

\[
dS_t = S_t (\mu dt + \sigma dW_t) ,
\]

where \((W_t)_{t \in [0,T]}\) is a standard Brownian motion. Furthermore, there is a risk-less bond with value \((B_t)_{t \in [0,T]}\) paying a return of \(r \leq \mu\) per unit of time

\[
dB_t = r B_t dt .
\]

Each fund-manager constantly rebalances her portfolio and decides which fraction \(\beta_t\) of her assets under management to invest in the risky asset and which in the riskless bond. While the financial mathematics literature mostly restricts the space of strategies such that the fund-manager can base her decision which assets to buy and which to sell at time \(t\) only on past prices \((S_s)_{s \leq t}\) we allow the fund-manager to randomly decide on her investments:

**Definition 2 (Mixed Strategy).** A mixed trading strategy \(\beta\) is a mapping from \([0, 1]\) into the space of square integrable processes adapted to the natural filtration of the Brownian motion \(W\).

Intuitively, when playing a mixed strategy a fund-manager first draws a random number \(\alpha \in [0, 1]\) from the uniform distribution (independently of everything else). The number \(\alpha\) then in combination with the past evolution of asset prices determine the trades. At each point in time \(t\) the fund-managers invests \(\beta_t(\alpha)\) of her assets under management into the risky asset and \(1 - \beta_t(\alpha)\) in bonds. For fixed \(\alpha\) the trades made at time \(t\) are only a function of past asset prices \((S_s)_{s \leq t}\) and thus the random number \(\alpha\) is the only way in which non-market related randomness enters the decision process of the fund-manager.\(^{13}\) The trades of fund-manager \(i\) do not depend on the trades of fund-manager \(j \neq i\) as this information is in general not available to competing fund-managers.\(^{14}\)

\(^{12}\)For a detailed discussion of this two asset model with consumption see for example Chapter 4.3 in Merton and Samuelson [1992] or Chapter 3.3 in Dana and Jeanblanc [2007].

\(^{13}\)A classical trading strategy as found in most finance textbooks is a mixed trading strategy independent of \(\alpha\).

\(^{14}\)For example the SEC website states that even “investors typically cannot ascertain the exact make-up of a fund’s portfolio at any given time” (see http://www.sec.gov/investor/pubs/inwsnf.htm).
The dynamics of the fund-managers portfolio return when she uses the trading strategy $\beta$ are given by\textsuperscript{15}

$$dX_t^{\beta,\alpha} = X_t^{\beta,\alpha} \left([r + \beta_t(\alpha)(\mu - r)] \, dt + \beta_t(\alpha)\sigma \, dW_t\right).$$

To show that the Black-Scholes model is a special case of our general market model we have to specify what states of the market are. Here we take values of the risky asset $s = S_T \in \mathbb{R}_+$ at time $T$ as states of the market. The physical measure over states then is the measure induced by the geometric Brownian motion $\rho(\cdot) = \mathbb{P}[S_T \in \cdot]$.

Furthermore, we have to define the pricing measure $\rho^*$ which is used to determine whether a portfolio return distribution can be generated using a trading strategy. Let us denote by $\mathbb{P}^*[\cdot]$ the measure under which the asset price process is a geometric Brownian motion with drift $r$, and by $\mathbb{E}^*[\cdot]$ the corresponding expectation operator. It is well known that a deterministic portfolio return $Z : \mathbb{R}_+ \to \mathbb{R}_+$ can be generated $X_t^{\beta,\alpha} = Z(s)$ using a strategy $\beta$ with assets under management of $X_t$ at time $t < T$ if and only if\textsuperscript{16}

$$e^{-r(T-t)}\mathbb{E}^*[Z(S_T) \mid S_t] = X_t.$$  

As deterministic portfolio returns are a special case of random portfolio returns it follows that $\rho^*(\cdot) = e^{-rT}\mathbb{P}^*[S_T \in \cdot]$ is the only candidate for the pricing measure and thus

$$\Gamma(s) = s \frac{\mu - r}{\sigma^2} \exp\left(\frac{1}{2}(\mu + r) \left(1 - \frac{\mu - r}{\sigma^2}\right) T\right).$$

While this establishes that the Black-Scholes model is complete in the classical sense random completeness as we defined it in Assumption 1 requires that also all random portfolio returns which satisfy Condition 2 can be replicated using a mixed trading strategy. The next theorem shows that this is indeed the case.

**Theorem 1.** *The Black-Scholes model with mixed trading strategies as is randomly complete.*

Let us give a brief intuition for the proof provided in the Appendix. Suppose that each fund-manager can exchange their initial assets under management for any arbitrary random variable with the same expected value. In this case she can generate every random return distribution $F(x, s) = \mathbb{P}[X_T \leq x \mid S_T = s]$ which has a price smaller one under the pricing measure $\rho^*$ in the following way: Let $\alpha$ be uniformly distributed on $[0, 1]$ independent from $S$. First, the fund-manager exchanges her initial endowment for the random variable $Z(\alpha)$ given by

$$Z(\alpha) = \int_{\mathbb{R}_+} F^{-1}(\alpha, s)d\rho^*(s).$$

\textsuperscript{15}cf Eq. 4.12 in Merton and Samuelson [1992].
\textsuperscript{16}Proposition 4.4.2 and Proposition 4.4.3 in Dana and Jeanblanc [2007].
By definition $F^{-1}(\alpha, s)$ has the correct distribution and as $\Pi(X) = 1$ the random variable $Z(\alpha)$ has an expected value of one. As for fixed $\alpha$ the payoff profile $s \mapsto F^{-1}(\alpha, s)$ is deterministic and thus there exists a pure self-financing strategy generating the payoff $F^{-1}(\alpha, s)$ as the Black-Scholes model is complete in the classical sense.

The second step of the proof shows that the trader can use the random movements of the value of the risky asset in a short time period $[0, \hat{t}]$ to replace her endowment with an arbitrary random variable independent of the asset prices. In order for such a random payoff to be self-financing its expectation needs to be independent of $\alpha$, while at the same time include randomness independent of the asset price process. We generate such endogenous randomness using only $\alpha$ and the randomness of the asset price process. The existence of such a new random variable that is pairwise independent of $\alpha$, $S$, but at the same time only a function of $\alpha$ and $S$ follows from an argument in cryptography that constructs such “correlation immune” functions using the XOR operator on the binary representation of uniform random variables.

3. EQUILIBRIUM CHARACTERIZATION

We now return to the abstract model introduced in Section 2, but stay with the interpretation of fund-managers and trading strategies given in section 2.1. Denote by $F^i : \mathbb{R}_+ \times \mathcal{S} \to [0, 1]$ the equilibrium probability that fund-manager $i$’s final portfolio value $X^i$ is less than $x$ conditional on the state $s$.

$$F^i(x, s) = \mathbb{P}^\beta[X^i \leq x \mid S = s]$$

Recall the definition of the set of feasible distributions $Q$, such that for each $F^i \in Q$ there exists a trading strategy such that the return conditional on the market outcome $s$ is distributed according to $F^i(\cdot, s)$

$$Q = \left\{ F : \int_S \int_{\mathbb{R}_+} (1 - F(x, s)) \, dx \, d\rho^*(s) \leq 1 \right\} .$$

As we restrict attention to symmetric equilibria we set $F^i = F^j = F$. Suppose that the equilibrium distribution $F$ admits a density and denote it by $f = \frac{\partial}{\partial x} F$.

17 As the distribution function $F$ is absolutely continuous the probability of a tie is zero. The expected compensation of fund-manager $i$ conditional on the state $S = s$ and her own outcome $X^i = x$ equals

$$\mathbb{E}^\beta [b_{R(i)} \mid X^i = x, S = s] = \sum_{k=1}^{n} b_k \mathbb{P}^\beta [R(i) = k \mid X^i = x, S = s]$$

$$= b_1 + \sum_{k=1}^{n-1} (b_{k+1} - b_k) \mathbb{P}^\beta [R(i) \geq k \mid X^i = x, S = s] .$$

17 We show this in the prove of Theorem 2.
As the equilibrium is symmetric it follows that conditional on the state $S$ and her own outcome $X_i$ the rank of manager $i$ is binomial distributed depending only on the equilibrium probability $F(x, s)$ that any one of her competitors performed worse than her

$$\mathbb{P}^\beta \left[ R(i) = k \mid X_T^i = x, S_T = s \right] = \binom{n-1}{k-1} F(x, s)^{k-1} (1 - F(x, s))^{n-k}.$$  

We define the function $\phi_b : [0, 1] \rightarrow [0, 1]$ as the expected normalized compensation fund-manager $i$ receives when the probability that a given competitor performed worse that her equals $y \in [0, 1]$

$$\phi_b(y) \triangleq \sum_{k=1}^{n} \frac{b_k - b_1}{b_n - b_1} \binom{n-1}{k-1} y^{k-1} (1 - y)^{n-k}. \tag{4}$$

The following lemma shows several useful properties of $\phi_b$:

**Lemma 1.** $\phi_b$ has the following properties:

i) $\phi_b$ is strictly increasing with $\phi_b(0) = 0, \phi_b(1) = 1$.

ii) $\phi_b$ is invariant under quasi linear transformation of the compensation $b$, i.e. $\phi_b(y) = \phi_{c_1+c_2} b(y)$ for all $b \in \mathbb{R}^n, c_1 \in \mathbb{R}, c_2 \in \mathbb{R}_+, y \in [0, 1]$.

iii) $\int_0^1 \phi_b(z) \, dz = \frac{b - b_1}{b_n - b_1}$.

The expected compensation of manager $i$ in any symmetric equilibrium can be represented using $\phi_b$ as

$$\mathbb{E}^\beta \left[ b_{R(i)} \mid X^i = x, S = s \right] = b_1 + (b_n - b_1) \phi_b(F(x, s)). \tag{5}$$

Thus, any best response maximizes the expectation of $\phi_b(F(x, s))$ over the set of feasible distributions $Q$. Furthermore, a symmetric equilibrium distribution of this game is given by a distribution function $F$ that is a best response to itself:

**Proposition 1.** A feasible trading strategy $\beta$ is a symmetric equilibrium strategy if and only if the cdf $F$ induced by $\beta$ satisfies

$$\max_{G \in Q} \int_S \int_{\mathbb{R}} \phi_b(F(x, s)) dG(x, s) d\rho(s) \leq \frac{\overline{b} - b_1}{b_n - b_1}. \tag{6}$$

**Proof.** If all fund-managers use the same strategy all of them win each compensation with equal probability of $1/n$. Thus, the payoff when all fund-managers use the same strategy is given by the average price $\overline{b}$. Suppose a fund-manager deviates to a strategy that leads to the distribution of returns $G \in Q$, by Eq. 5 her expected payoff is given by

$$b_1 + (b_n - b_1) \left[ \int_S \int_{\mathbb{R}} \phi_b(F(x, s)) dG(x, s) d\rho(s) \right]. \tag{7}$$

Rearranging yields that (7) is smaller $\overline{b}$, and thus no deviation is profitable, if and only if (6) is satisfied. \hfill \square
Both the left and the right-hand side of (6) are invariant under linear transformations of the compensation structure. Intuitively, shifting or scaling all prizes by a constant is irrelevant for the agents incentives and does neither change her behavior nor the set of equilibria.

The characterization of equilibrium strategies in terms of induced distributions given in Lemma 1 shows that to check whether a strategy is an equilibrium strategy it suffices to characterize the distribution over returns it induces. It also implies that if there exist multiple strategies which lead to a given equilibrium distribution all of them are equilibrium strategies. Thus, while the model allows us to predicts the distribution of returns generated in equilibrium it in general does not uniquely predict the trading strategies fund-managers use to arrive at this distribution.

The next theorem shows that the equilibrium distribution of fund returns is unique:

**Theorem 2 (Equilibrium Characterization).** The unique symmetric equilibrium distribution of the game \( F_b : \mathbb{R}_+ \times S \to [0, 1] \) is given by

\[
F_b(x, s) = \begin{cases} 
\phi_b^{-1} \left( \frac{x}{\overline{x}(s)} \right) & \text{for all } x \in [0, \overline{x}(s)], \\
1 & \text{otherwise}
\end{cases}
\]

with the right end-point of the distribution \( \overline{x}_b \) given by \( \overline{x}_b(s) = \Gamma(s) \frac{b_n - b_1}{b - b_1} \). In equilibrium, the expected return of a fund conditional on the state of the market \( S = s \) is given by \( \Gamma(s) \)

\[
\mathbb{E}^\beta [X \mid S = s] = \Gamma(s).
\]

The proof of Theorem 2 given in the appendix proceeds as follows: First, plugging \( F_b \) in Equation 6 yields that any strategy such that the portfolio return \( X \) is in \([0, \overline{x}_b(s)]\) with probability one is a best response to \( F_b \). Thus, \( F_b \) is a best response to itself and an equilibrium distribution. In this equilibrium each fund-manager is indifferent between all trading strategies such that

\[ X_T \in [0, \overline{x}_b(S_T)] \]

with probability one and hence can randomize in such a way that she generates the distribution \( F_b \) over portfolio returns.

We show uniqueness of the equilibrium distribution \( F_b \) in Corollary 3 in Section 7 by considering an auxiliary game where each fund-manager can invest in her own fund at a fixed linear cost and is not restricted to feasible distributions. With appropriately chosen costs this problem is the Lagrangian of the original problem and each equilibrium in the original game corresponds to an equilibrium in the auxiliary game. As the feasibility constrained is not present in the auxiliary game it can be decomposed into independent games where each player chooses a strategy for each state \( s \) independently. Those auxiliary games are all-pay contests with an arbitrary prize structure, for which we show in Proposition 7 that they admit a unique symmetric equilibrium.
4. Welfare Analysis

This section analyzes welfare consequences of the equilibrium investment behavior of fund-managers. We assume that the money managed by fund-managers comes from small risk-averse investors. Those investors are risk averse in the sense that they prefer the random outcome $X$ over the random outcome $Y$ when $X$ dominates $Y$ in the sense of second order stochastic dominance. Many preferences satisfy risk-aversion in the above sense, for example expected utility with concave utility function.

4.1. The Equilibrium Outcome is Inefficient. To make statements about welfare we need to compare the portfolio outcome induced by equilibrium behavior of fund-managers to the outcome induced by some alternative trading strategy. We first focus on the strategy $\hat{\beta}$ which an expected utility maximizer with constant relative risk aversion of one would chose:

$$\hat{\beta} \in \arg \max_{\beta} E[\beta \log(X_T)].$$

The objective (10) is known as the Kelly criterion (Kelly [1956]). Kelly proposed the criterion as $\hat{\beta}$ maximizes the long-run rate of return.

Our first result shows that the return distribution induced by the Kelly strategy $\hat{\beta}$ dominates the return distribution induced by the equilibrium behavior of fund-managers.

**Proposition 2** (Inefficiency of the equilibrium outcome). *Every (strictly) risk-averse investor (strictly) prefers the distribution of returns under the trading strategy $\hat{\beta}$ to the distribution of returns under any trading strategy used in any market equilibrium.*

Proposition 2 holds as the Kelly strategy yields the same expected return conditional on the state as the equilibrium strategy

$$E^{\hat{\beta}}[X_T \mid S_T = s] = E^{\beta}[X_T \mid S_T = s],$$

but in equilibrium fund-managers introduce non-market related randomness into their portfolio returns. As show in the proof of Proposition 2 in the appendix they do so in such a way that the returns under the Kelly strategy dominate the equilibrium returns in the sense of second order stochastic dominance.

It is worth highlighting that the risk created by the fund-managers in equilibrium is entirely unrelated to the risk coming from the underlying financial market. Fund-managers introduce this risk by making random portfolio choices in order to be unpredictable to their peers. Risk-averse investors are hurt by this endogenously and inefficiently introduced risk.

Proposition 2 holds for any market structure: for an arbitrary number of assets, an arbitrary expected return, variance and correlation between those assets and in discrete time as well as
continuous time models. In that sense, the inefficiency of the equilibrium outcome is a robust result.

In the next step we quantify the amount of endogenous risk fund-managers create in equilibrium. As the return of the Kelly strategy depends only on the market prices of the different assets we denote by $X^β(s)$ the return the Kelly strategy yields in market state $s$. To quantify the amount of endogenous risk fund-managers create in equilibrium we define the standard deviation of equilibrium returns conditional on the market state

$$\text{SD}^β[X, s] = \sqrt{\mathbb{E}^β[X^2 | S = s] - \mathbb{E}^β[X | S = s]^2}.$$ 

The standard deviation is arguably one of the simplest descriptive statistics to measure the endogenous risk created in equilibrium.

**Proposition 3.** The standard deviation of equilibrium returns conditional on the market state is given by

$$\text{SD}^β[X, s] = X^β(s) R_b$$

where $R_b$ is independent of the market structure and given by

$$R_b = \sqrt{\int_0^1 \left[ \sum_{k=1}^n \frac{b_k - b_1}{b - b_1} \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} \right]^2 dy - 1}.$$ 

A remarkable implication of Proposition 3 is that for every realization of market prices $s$ the standard deviation of fund returns equals the return of the Kelly Strategy times a constant $R_b$ which is completely independent of the specification of the underlying financial market. In that sense the amount of endogenous risk created in equilibrium depends mostly on the incentives fund-managers face. The financial market in which they interact influences the amount of risk created in equilibrium only through the return of the Kelly strategy $X^β(s)$. An immediate corollary is the following:

**Corollary 1.** The amount of the endogenous risk created by fund-managers in equilibrium (as measured by its standard deviation) is higher when the return of the Kelly strategy is higher.

A testable implication is that the returns of different funds should differ more in times when market returns are high compared to times when market returns are low.

The constant $R_b > 0$ describes how much endogenous risk fund-managers create. $R_b$ is independent of the structure of the underlying financial market and easy to compute given an estimate of
the compensation structure fund-managers face. For example for the linear compensations structure \( b_k = (k - 1) \) we have that

\[
R_b = \sqrt{\int_0^1 \left[ \sum_{k=1}^n \frac{b_k - b_1}{b - b_1} y^{k-1}(1 - y)^{n-k} \right]^2 dy} - 1 = \sqrt{2^2 \int_0^1 y^2 dy} - 1 = \sqrt{1/3}.
\]

This implies that the standard deviation of the equilibrium return conditional on the state is \( \sqrt{1/3} \approx 58\% \) of 1 plus the expected return in that state. If the Kelly strategy realizes a return of 5\% this implies that the standard deviation of fund returns is \( \sqrt{1/3} \times 1.05 \approx 68\% \). This means, that the endogenous risk created in equilibrium – which is entirely unrelated to market risk – has the same standard deviation as a gamble in which the investors loose 68\% of their money with probability 1/2 and win an additional 68\% with probability 1/2.

Proposition 3 also allows us to characterize the unconditional standard deviation of the funds equilibrium returns:

**Proposition 4.** The standard deviation of of equilibrium returns is given by

\[
(13) \quad \text{SD}\beta^*[X] = \sqrt{(R_b^2 + 1) \text{SD}\beta^*[X]^2 + R_b^2 \mathbb{E}\beta^*[X]^2}.
\]

Proposition 4 characterizes shows that the standard deviation of the equilibrium return distribution is only a function of \( R_b \), the standard deviation of the Kelly strategy and the expected return of the Kelly strategy. To illustrate this result we calculate standard deviations and returns for the Kelly strategy and the equilibrium strategy in an example with a Black-Scholes financial market in Section 6, Table 1.

5. Comparative Statics

In this section we show how changes in the compensation structure \( b \) influence the risk-taking behavior of fund-managers. The first result relates the distribution of outcomes \( X_b(s) \) conditional on the market state \( s \) to the compensation structure fund-managers face:

**Proposition 5** (Equilibrium Efficiency). The equilibrium distribution with compensation structure \( b \in \mathbb{R}^n \) and \( n \) fund-managers second order stochastically dominates the equilibrium return distribution with compensation structure \( \tilde{b} \in \mathbb{R}^m \) and \( m \) competing fund-managers if and only if for all \( \alpha \in [0, 1] \)

\[
(14) \quad \frac{b_n - b_1}{b - b_1} \int_0^\alpha \phi_b(y)dy \geq \frac{\tilde{b}_m - \tilde{b}_1}{\tilde{b} - \tilde{b}_1} \int_0^\alpha \phi_{\tilde{b}}(y)dy.
\]
Proposition 5 induces an partial order over compensation schemes. If two compensations vectors \( \mathbf{b}, \tilde{\mathbf{b}} \) are ordered according to Eq. 14 it implies that all risk-averse investors order the resulting distribution of portfolio returns the same way.\(^\text{18}\)

5.0.1. Increasing the Compensation of High-Ranked Fund-Managers relative to Low Ranked Fund Managers. Suppose that investors become more likely to pick a fund that performed well in the past. This can be interpreted as an increase in competition between the fund-managers as high ranked fund-managers now capture a larger part of the market. We say that a compensation structure becomes more competitive if the compensation received by the \( k \)-th ranked manager is decreased by \( \Delta > 0 \) while at the same time the compensation for the \( j > k > 1 \) ranked fund-manager is increased by the same amount and the remaining compensations remain unchanged, i.e.

\[
\begin{align*}
\tilde{b}_k &= b_k - \Delta \\
\tilde{b}_j &= b_j + \Delta \\
\tilde{b}_i &= b_i \quad \text{for all } i \notin \{k, j\}.
\end{align*}
\]

We call a compensation structure more competitive than another compensation structure if it can be generated by iteratively shifting mass upwards in the way described above.\(^\text{19}\)

**Theorem 3.** If the compensation structure \( \mathbf{b} \in \mathbb{R}^n \) is more competitive than \( \tilde{\mathbf{b}} \in \mathbb{R}^n \) then the equilibrium distribution \( F_b(\cdot, s) \) is second order stochastically dominated by \( F_{\tilde{b}}(\cdot, s) \) for every \( s \in S \).

An immediate implication of Theorem 3 which is proven in the appendix is that every risk-averse investors prefers the equilibrium distribution of returns under the compensation structure \( \tilde{\mathbf{b}} \) over the equilibrium distribution of returns under the more competitive compensation structure \( \mathbf{b} \). Furthermore, it follows that the compensation structure which induces the most risk is the one where only the best performing fund-manager wins a prize

\[ \mathbf{b} = (0, \ldots, 0, 1) \]

\(^{18}\)While the efficiency order \( \preceq \) implied by Eq. 14 is partial for more than four fund-managers the order is complete for situations where three or less fund-managers compete \( n \leq 3 \). Simple algebra shows that i) \((b_1, b_2) \preceq (\tilde{b}_1, \tilde{b}_2)\) for all \( \mathbf{b}, \tilde{\mathbf{b}} \in \mathbb{R}^2 \) ii) for all \( \mathbf{b} \in \mathbb{R}^2, \tilde{\mathbf{b}} \in \mathbb{R}^3 \) we have \((b_1, b_2) \preceq (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)\) if and only if \( \frac{b_2 - b_3}{b_2 - \tilde{b}_3} \geq \frac{\tilde{b}_2 - \tilde{b}_3}{\tilde{b}_2 - b_3} \) iii) for all \( \mathbf{b}, \tilde{\mathbf{b}} \in \mathbb{R}^3 \) we have \( \mathbf{b} \preceq \tilde{\mathbf{b}} \) if and only if \( \frac{b_2 - b_3}{b_2 - b_1} \geq \frac{\tilde{b}_2 - \tilde{b}_3}{\tilde{b}_2 - \tilde{b}_1} \).

\(^{19}\)It is well known that \( \tilde{\mathbf{b}} \) can be generated by such upwards shifts from \( \mathbf{b} \) if and only if it majorizes \( \mathbf{b} \), i.e. if for all \( \sum_{r=1}^m b_r \leq \sum_{r=1}^m \tilde{b}_r \) for all \( m \in N \), see for example Arnold [2012, page 11]. Hence, the normalized prize vector corresponding to \( \tilde{b} \) is more competitive than \( b \) if and only if \( \sum_{r=1}^m \frac{b_r - b_1}{\tilde{b}_r - b_1} \leq \sum_{r=1}^m \frac{b_r - b_1}{\tilde{b}_r - b_1} \).
and that the prize structure which induces the least risk is the one where all, but the worst performing fund manager win a prize

\[ b = (0, 1 \ldots, 1). \]

**Corollary 2.** The equilibrium distribution \( F_b(\cdot, s) \) with the compensation structure \( b \in \mathbb{R}^n \)

1. is second order stochastically dominated by \( F_{b'}(\cdot, s) \) with \( b' = (0, 1, \ldots, 1) \in \mathbb{R}^n \) and
2. second order stochastically dominates \( F_{b''}(\cdot, s) \) with \( b'' = (0, \ldots, 0, 1) \in \mathbb{R}^n \)

for all \( s \in S \).

The second part of Corollary 2 is similar to Moldovanu and Sela [2001] who find that the compensation structure which maximizes effort in an all-pay contest with privately known costs is an all-pay auction where only the highest bidder wins a prize.

5.1. **Increasing the Number of Fund-Managers.** Starting from a situation with \( n \) fund-managers, we introduce \( m > 1 \) additional fund-managers and keep the compensation scheme fixed. As we kept the sum of prizes fixed, but increase the number of competitors this can be interpreted as an increase in competition. Formally, denote by \( \tilde{b} \in \mathbb{R}^{n+m} \) the compensation vector where \( m + n \) fund-managers compete and the \( m \) worst performing fund-managers receive the payoff \( b_1 \) equal to the worst payoff in the original compensation scheme and the \( n \) best performing fund-managers receive the original compensation \( b \), ie.

\[
\tilde{b}_k \triangleq \begin{cases} 
  b_1 & \text{for } k \leq m \\
  b_{k-m} & \text{for } k > m .
\end{cases}
\]

We have the following result showing that increasing competition by having more fund-managers compete for the same compensations makes the equilibrium return distribution more risky:

**Theorem 4.** The equilibrium distribution \( F_{\tilde{b}}(\cdot, s) \) with the compensation structure \( \tilde{b} \in \mathbb{R}^n \) second order stochastically dominates the return distribution \( F_{\tilde{b}}(\cdot, s) \) with \( \tilde{b}_k = b_{\max\{k-m,1\}} \in \mathbb{R}^{n+m} \) for every \( s \in S \).

Hence, every risk averse investor prefers the portfolio outcome in any equilibrium with \( n \) fund-managers competing for the compensations \( b \) to the outcome in any equilibrium with \( n + m \) fund-managers and the compensation structure \( \tilde{b} \). As the proof of Theorem 4 in the Appendix shows fund-managers react to the increase in competition by choosing more risky investment strategies. However, by Eq. 9 of Theorem 2 the expected return conditional on the state is independent of the compensation structure and the number of fund-managers. Thus, somewhat surprisingly, fund-managers do not increase the riskiness of their portfolio by shifting investments towards assets with higher return and higher risk. Instead, they make their portfolio decisions more random which create more endogenous risk, but does not lead to higher expected returns. The reason for this
behavior is that fund-managers in equilibrium do not aim at increasing expected return, but their expected (appropriately weighted) rank. If more fund-managers compete the expected weighted rank conditional on the state becomes more convex and thus fund-managers take on more state independent, i.e. non-market risk. Consequently, even though investors face a more risky portfolio return if competition increases their expected return does not increase and thus competition makes every risk-averse investor worse off.

Theorem 4 assumes that the worst performing fund-managers receive compensation equal to the lowest original compensation \( b_1 \). Not paying out any compensation to the worst performing fund-managers can revert the result and lead to an improvement in the equilibrium return distribution in the sense of second order stochastic dominance.\(^{20}\) The reason for this is that fund-managers evaluate compensation relative to the minimal compensation and thus decreasing minimal compensation can reduce incentives to take on risk.

6. A Black Scholes Example

6.1. Financial Market. This section illustrates the equilibrium in the context of the Black Scholes model we introduced in Section 2.3. We calibrate the model such that the risk-less interest rate equals two percent \( r = \log(1.02) \) and the expected return of the risky asset is between 2 and 7 percent \( \mu \in [\log(1.02), \log(1.07)] \). We choose the volatility \( \sigma \) such that the standard deviation of the yearly return of the risk asset equals 18 percent.\(^{21}\) This matches roughly the standard deviation of the empirical returns of the Dow Jones Industrial Average index between 1898 and 2015.\(^{22}\) The next result shows that in the Black-Scholes model the Kelly Strategy is of a particularly simple form: \(^{23}\)

**Proposition 6** (Kelly Strategy in the Black Scholes Model). The Kelly strategy invests a fixed fraction \( \frac{\mu - r}{\sigma^2} \) of the wealth at every point in time in the risky asset.

6.2. Compensation Structure. The compensation structure used for the calculation of equilibrium is taken from Reuter and Zitzewitz [2010]. Reuter and Zitzewitz estimate the influence the number of stars in the morning star rating has on the in and outflows of investors money into a

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\(^{20}\) Too construct such an example consider the situation with two fund-managers where \( b = (\frac{3}{7}, \frac{4}{7}) \) adding a third fund-manager and changing the compensation scheme to \( \tilde{b} = (0, \frac{3}{7}, \frac{4}{7}) \). Simple algebra shows that in this case \( F_b^{-1}(y, s) - \Gamma(s) 2/7y(3y - 2) \), as is negative for \( y < 2/3 \) and positive for \( y > 2/3 \) it follows that \( F_b(\cdot, s) \) is second order stochastically dominated by \( F_{\tilde{b}}(\cdot, s) \). Hence, the equilibrium return distribution \( F_b \) is preferred by every risk-averse investor over \( F_{\tilde{b}} \).

\(^{21}\) If the standard deviation \( \sqrt{\mathbb{E}[S_T^2] - \mathbb{E}[S_1]^2} \) of the return of the risky asset at time \( T \) equals \( y > 0 \) then the diffusion coefficient of the asset price process \( S \) in the Black Scholes models is given by \( \sigma = \frac{1}{\sqrt{T}} \sqrt{\log (e^{2\mu T} + y^2) - 2\mu T} \).


\(^{23}\) This result is well known. We provide a short proof in the appendix as no version of the result we could find in a textbook matched exactly the setting here.
### Table 1. Expected return and standard deviation of the return for different investments assuming fifty competing fund-managers.

Here we assume that fund-managers maximize the empirical growth of their fund as a function of its Morningstar rating as estimated in Reuter and Zitzewitz [2010]. The underlying financial market is modeled by a Black-Scholes model with $T = 1$ and different parameters.

<table>
<thead>
<tr>
<th>Bond Return</th>
<th>Risk Asset Return</th>
<th>Funds Strategy Return</th>
<th>Index Strategy Return</th>
<th>$\beta^i$</th>
<th>Return</th>
<th>StdDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>2%</td>
<td>2%</td>
<td>2%</td>
<td>0%</td>
<td>2%</td>
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<tr>
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<td>3%</td>
<td>2.3%</td>
<td>32.4%</td>
<td>2.3%</td>
<td>5.8%</td>
<td></td>
</tr>
<tr>
<td>2%</td>
<td>4%</td>
<td>3.3%</td>
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<td>3.3%</td>
<td>11.7%</td>
<td></td>
</tr>
<tr>
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<td>5%</td>
<td>5%</td>
<td>100.1%</td>
<td>5%</td>
<td>18%</td>
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</tr>
<tr>
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<td>6%</td>
<td>7.4%</td>
<td>135.3%</td>
<td>7.4%</td>
<td>24.8%</td>
<td></td>
</tr>
<tr>
<td>2%</td>
<td>7%</td>
<td>10.7%</td>
<td>171.5%</td>
<td>10.7%</td>
<td>32.4%</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Expected return and standard deviation of the return for different investments assuming fifty competing fund-managers. Here we assume that fund-managers maximize the empirical growth of their fund as a function of its Morningstar rating as estimated in Reuter and Zitzewitz [2010]. The underlying financial market is modeled by a Black-Scholes model with $T = 1$ and different parameters.

As the Morningstar ranking is mainly based on relative performance of a fund within a category\(^{24}\) we get that the compensation structure of fund-managers under the assumption that they aim at myopically maximizing their relative inflows every year is given by

$$b_i = \begin{cases} 
+16.1 & \text{if } \frac{R(i)}{n} \in [0.9, 1.0] \\
+5.9 & \text{if } \frac{R(i)}{n} \in [0.675, 0.9) \\
-2.0 & \text{if } \frac{R(i)}{n} \in [0.325, 0.675) \\
-8.1 & \text{if } \frac{R(i)}{n} \in [0.1, 0.325) \\
-12 & \text{if } \frac{R(i)}{n} \in [0, 0.1) 
\end{cases}$$

6.3. **Equilibrium Predictions.** Table 1 shows the return and standard deviation of returns for the Kelly strategy and the equilibrium trading strategy with fifty fund-managers. A first observation is that (as derived in Proposition 2) the equilibrium strategy leads to a distribution of returns which has the same expected value as the distribution induced by the Kelly strategy, but to a substantially higher risk. In this example the standard deviation of the returns is between 22% and 34% higher when investing in a fund compared to using a Kelly strategy.

This additional risk is most striking in the case where both the bond as well as the risky asset yield an (expected) return of 2%. In this case there is no motive to buy the risky asset as it increases

\(^{24}\)“Morningstar rates mutual funds from one to five stars based on how well they’ve performed (after adjusting for risk and accounting for all sales charges) in comparison to similar funds. Within each Morningstar Category, the top 10% of funds receive five stars, the next 22.5% four stars, the middle 35% three stars, the next 22.5% two stars, and the bottom 10% receive one star.” see [http://screen.morningstar.com/FundInsightsDefinitions/RiskRating2.html](http://screen.morningstar.com/FundInsightsDefinitions/RiskRating2.html)
the variance of the portfolio while not increasing the expected return. Consequently, the Kelly strategy invests all wealth in the risk-less bond ($\hat{\beta}_t = 0$ for all $t \in [0, T]$) and its return distribution has a standard deviation of zero. In sharp contrast fund-managers in equilibrium invest into the risky asset even though it yields the same expected return as the risk-less bond. Furthermore, by randomly trading the risky asset they create a portfolio return which has a 3.7 higher standard deviation than the return from just holding the risky asset. This shows that the endogenous risk created in equilibrium can be multiple times greater than the risk coming from the underlying financial market and thus has a first order effect on investors welfare.

If one increases the expected return of the risky asset both the Kelly strategy and the equilibrium strategy used by fund-managers invests more in the risky asset. For expected returns higher than five percent both strategies start leveraging the assets under management to generate a return greater than the return of the risky asset. For example, if the expected return of the risky asset equals six percent the Kelly strategy will hold debt which values at 35.3% of the current portfolio value.

In the Supplementary Appendix we verify show how the results are to changes in the compensation structure. Table 2 shows the results for a linear compensation structure, Table 3 shows the results for the maximally competitive compensation structure and Table 4 shows the results for the maximally competitive compensation structure.

7. Relation to All-Pay Contests

In this section we explore how the general model introduced at the beginning of Section 2 relates to symmetric full-information all-pay contests. In an all-pay contest each agent $i$ chooses a score $X^i \in \mathbb{R}_+$ and receives a prize $b_{R(i)}$ which is only a function of her rank $R(i)$ and pays a cost which is linear in her score $c X^i$. As before we assume that ties are broken randomly with equal probability. A Bayesian equilibrium of an all-pay contest is a vector of distributions $F^i_b : \mathbb{R}_+ \rightarrow [0, 1]$ such that

$$F^i_b \in \arg \max_{G} \mathbb{E}^{(G,F^i_b)} \left[ b_{R(i)} - c X^i \right].$$

As before we consider only symmetric equilibria $F_b = F^i_b = F^j_b$ for all $i, j \in N$. As it was shown in Barut and Kovenock [1998, Theorem 2 and Theorem 3] the unique symmetric equilibrium of this all-pay contest is given by\textsuperscript{25}:

**Proposition 7** (Barut and Kovenock [1998]). The all-pay contest with $n$ bidders, linear cost $c x$ and compensations $b_1 \leq b_2 \leq \ldots \leq b_n$ has a unique symmetric equilibrium

$$F_b(x) = \frac{1}{\phi_b} \left( \frac{c x}{b_n - b_1} \right).$$

\textsuperscript{25}For the convenience of the reader we provide a self-contained proof in the Appendix.
Furthermore, the mean equilibrium bid $E^F[X^i] = \frac{b - b_1}{c}$ is strictly decreasing in $c$.

Let $m = E^F[X^i] = \frac{b - b_1}{c}$ expected score in equilibrium. If we restrict the agents to choose distributions from the set

$$Q \triangleq \{G : \mathbb{R}_+ \rightarrow [0, 1] : E^G[X^i] \leq m\}$$

the vector of distributions $(F_b, \ldots, F_b)$ remains an equilibrium as we reduced the set of potentially profitable deviations. Intuitively, we introduced a non-binding budget constraint into the contest game. Because the compensation is monotone in the bid no agent will chose a distribution $G$ in equilibrium for which $E^G[X^i] < m$ and we can thus wlog restrict to distributions such that $E^G[X^i] = m$. As for this set of distributions the cost an agent pays are fixed they do not play any role for the optimality constrained. If we define the set of states $S = \{s\}$ to be a singleton and define $\rho^*(\{s\}) = \frac{1}{m}$ we thus get that the equilibrium of the all-pay contest is an equilibrium of the contest model specified in Section 2:

**Proposition 8.** Every symmetric equilibrium of the all-pay contest with compensation structure $b$ is an equilibrium distribution of the contest game specified in Section 2 with $S = \{s\}, \rho(\{s\}) = 1, \rho^*(\{s\}) = \frac{c}{b - b_1}$.

Hence, when restricting attention to symmetric equilibria the all-pay contest is a special case of the general model specified in Section 2. The next result shows that also the opposite is true:

**Proposition 9.** Let $F_b : \mathbb{R}_+ \times S \mapsto [0, 1]$ be a symmetric equilibrium distribution of the contest game specified in Section 2 with state space $S$ and $\rho, \rho^* \in \Delta(S)$. For every $s$ the distribution $F^*(\cdot, s)$ is an equilibrium of the all-pay contest with compensation structure $b$ and state dependent cost

$$c(s) = \frac{(b - b_1)\Gamma^{-1}(s)}{\int_S \Gamma(s)^{-1}d\rho^*(s)}.$$

As a consequence of Proposition 8 and Proposition 9 there is a one-to-one relationship between symmetric equilibria in the general contest game introduced in Section 2 and the all-pay contest discussed in this section. As the symmetric equilibrium in the all-pay contest is unique by Proposition 7 it follows that the symmetric equilibrium derived in Theorem 2 is the unique symmetric equilibrium distribution:

**Corollary 3.** The equilibrium distribution given in Theorem 2 is the unique symmetric equilibrium distribution.

Furthermore, the isomorphism between equilibria in the all-pay contest and the equilibria in the game where fund-managers compete allows us to interpret the competition between fund-managers as a standard contest game.
Note, that the welfare implications of the comparative statics derived in Section 5 reverse in this application. While the contest designer is an investor in the managed fund application and naturally risk-averse we would expect the contest designer to be risk-loving in many standard contest situations. For example in an R&D contest the designer might aim at maximizing the highest bid of the contestants $\max\{X_1, \ldots, X^n\}$ which is a convex objective function.

While we found in Section 5 that competition, i.e. increasing the number of agents or allocating higher prizes to high ranked agents decreases welfare if the objective is concave the opposite is true for convex objectives:

**Corollary 4.** Increasing competition in the sense specified in Proposition 3 or Proposition 4 increases the welfare of the contest designer whenever his objective is convex.

This comparative static in the number of agents and the prize structure is to the best of my knowledge novel to the literature on all-pay contests.

8. Conclusion

This paper analyzed risk-taking behavior in contests with general compensation structures. With the competition between fund-managers in a dynamic Black-Scholes model as a special case. We find that fund-managers create risk unrelated to the underlying financial market by randomizing over investment decisions. This risk-hurts risk-averse investors and increases the standard deviation of returns substantially. Increasing competition between fund-managers leads them to pursue more risky investment strategies and thus further decreases investors welfare.

References


9. Appendix

Proof of Theorem 1: Let $\alpha$ be uniformly distributed on $[0, 1]$, independent of $(S_t)_{t \in [0, 1]}$. Fix a time $0 < \hat{t} < T$. We are going to construct a function $h : [0, 1] \times \mathbb{R}_+ \rightarrow [0, 1]$ such that $h(\alpha, S_{\hat{t}})$ is uniformly distributed on $[0, 1]$ pairwise independent of $S_{\hat{t}}$ and $\alpha$.

First, we define $y : \mathbb{R}_+ \rightarrow [0, 1]$ as the quantile of the distribution of $S_{\hat{t}}$, i.e. $\mathbb{P}[y(S_{\hat{t}}) \leq z] = z$. By construction $y(S_{\hat{t}})$ is uniformly distributed on $[0, 1]$, but not independent of $S_{\hat{t}}$.

Second, we construct a function $g : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that $g(\alpha, \tilde{\alpha})$ uniformly distributed on $[0, 1]$ independent of $\alpha$, $\tilde{\alpha}$ if $\alpha$ and $\tilde{\alpha}$ are independent uniform random variables on $[0, 1]$. Each number $\alpha \in [0, 1]$ has a unique representation $(k_j(\alpha))_{j \in \mathbb{N}}$ as a sequence of zeros and ones

$$\alpha = \sum_{j=1}^{\infty} 2^{-j} k_j(\alpha).$$

As $\alpha$ is uniform distributed on $[0, 1]$ it follows that

$$\mathbb{P}[k_j(\alpha) = 1] = \mathbb{P}[k_j(\alpha) = 0] = \frac{1}{2}$$

and further more all entries are independent. As $\alpha$ and $\tilde{\alpha}$ are independent it thus follows that

$$k_j(\alpha) \oplus k_j(\tilde{\alpha}) \triangleq \begin{cases} 1 & \text{if } k_j(\alpha) = k_j(\tilde{\alpha}) \\ 0 & \text{if } k_j(\alpha) \neq k_j(\tilde{\alpha}) \end{cases}$$

equals 0 with probability one-half and is pairwise independent of $\alpha$ and $\tilde{\alpha}$.\footnote{This resembles the construction of public randomization through communication in multi-agent repeated games.}

We can hence define

$$h(\alpha, S_{\hat{t}}) \triangleq \sum_{j=1}^{\infty} 2^{-j} (k_j(\alpha) \oplus k_j(g(S_{\hat{t}})))$$
and observe that $h(\alpha, S_{\hat{t}})$ is uniformly distributed on $[0, 1]$ and pairwise independent of $\alpha$ and $S_{\hat{t}}$.

Finally, we define a function $q : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$

$$q(\alpha, S_{\hat{t}}) = e^{-r(T-\hat{t})} \mathbb{E}^Q \left[ F^{-1}(h(\alpha, S_{\hat{t}}), S_T) \mid S_{\hat{t}} \right].$$

First, note that for every fixed $\alpha \in [0, 1]$ there exists a pure self-financing trading strategy that generates the payoff $q(\alpha, S_{\hat{t}})$. Too see this observe that $h(\alpha, S_{\hat{t}})$ is by construction independent of
\[ e^{-rT} E^Q \left[ q(\alpha, S_t) \mid \alpha \right] = e^{-rT} E^Q \left[ E^Q \left[ F^{-1}(h(\alpha, S_t), S_T) \mid S_t \right] \mid \alpha \right] \]
\[ = e^{-rT} E^Q \left[ \int_0^1 F^{-1}(\gamma, S_T) \, d\gamma \mid S_t \right] \]
\[ = e^{-rT} E^Q \left[ \int_0^1 F^{-1}(\gamma, S_T) \, d\gamma \right] \]
\[ = \int_{\mathbb{R}_+} E \left[ \int_0^1 F^{-1}(\gamma, S_T) \, d\gamma \mid S_T = s \right] \, dp^*(s). \]

Hence, we have shown that for every \( \alpha \) there exists a pure self-financing trading strategy which generates a payoff of \( q(\alpha, S_t) \). By definition of \( q \) there than also exists a pure self-financing trading strategy that generates the payoff \( F^{-1}(h(\alpha, S_t), S_T) \) at time \( T \). As \( h(\alpha, S_t) \) is uniformly distributed on \([0, 1]\) independent of \( S \) we have thus constructed a mixed self-financing strategy that generates a (random) payoff with distribution \( F \) and thus the Black-Scholes model is randomly complete. \( \square \)

**Proof of Lemma 1.** \( \phi_b(0) = 0 \) and \( \phi(1) = 1 \) is immediate from the definition. The strict monotonicity of \( \phi_b \) follows immediately as \( b_{k+1} - b_k > 0 \) for at least one \( k \). Note that \( \int_0^1 \phi_b(z) \, dz \) is the expected scaled price of a fund-manager whose probability of outperforming a competitor is randomly drawn from the uniform distribution and hence equals the scaled average price \( \overline{b} \)
\[ \int_0^1 \phi_b(z) \, dz = \frac{1}{b_n - b_1} \left( \sum_{k=1}^n b_k \frac{(n-1)}{k-1} \right) \int_0^1 z^{k-1}(1-z)^{n-k} \, dz - b_1 \]
\[ = \frac{1}{b_n - b_1} \left( \sum_{k=1}^n b_k \frac{(n-1)!}{(k-1)!} \frac{(k-1)!(n-k)!}{n!} - b_1 \right) \]
\[ = \frac{1}{b_n - b_1} \left( \frac{1}{n} \sum_{k=1}^n b_k - b_1 \right) = \frac{\overline{b} - b_1}{b_n - b_1}. \quad \square \]

**Proof of Theorem 2.** In the first step we prove that the distribution \( F_b \) leads to an expected return of \( \Gamma(s) \) in the market state \( s \). For every state \( s \) the expected return conditional on the state is given by \( \Gamma(s) \)
\[ \mathbb{E} \left[ X \mid S = s \right] = \int_0^{\overline{x}_b(s)} (1 - F_b(x, s)) \, dx = \int_0^{\overline{x}_b(s)} 1 - \phi_b^{-1} \left( \frac{x}{\overline{x}_b(s)} \right) \, dx = \overline{x}_b(s) \int_0^1 1 - \phi_b^{-1}(z) \, dz \]
\[ = \overline{x}_b(s) \int_0^1 (1 - z) \phi_b'(z) \, dz = \overline{x}_b(s) \int_0^1 \phi_b(z) \, dz = \Gamma(s). \]
Here we used that \( \int_0^1 \phi_b(z) \, dz = \frac{\overline{b} - b_1}{b_n - b_1} \) as shown in Lemma 1.
In the second step we show that the distribution $F_b$ is feasible, i.e. $F_b \in Q$. Integrating over states gives that the distribution $F_b$ is feasible

$$\int_S \mathbb{E}[X \mid S = s] \, d\rho^*(s) = \int_S \Gamma(s) \, d\rho^*(s) = \int_S \rho(s) \, ds = 1.$$ 

In the third step we prove that $F_b$ is a best reply to the other agents playing a strategy inducing the distribution $F_b$. By Proposition 1 we need to verify that no feasible distribution $G \in Q$ leads to a payoff higher than

$$\bar{b} \sum_{s} \min \left\{ \frac{x}{T_b(s)}, 1 \right\} dG(x, s) d\rho(s) \leq \bar{b}.$$

Plugging in the definition of $F_b$ yields

$$\bar{b} \sum_{s} \min \left\{ \frac{x}{T_b(s)}, 1 \right\} dG(x, s) d\rho(s) \leq \bar{b} - \frac{b_1}{b_n - b_1} \int_S \mathbb{E}[X \mid S = s] \frac{1}{\Gamma(s)} \, d\rho(s)$$

$$= \bar{b} - \frac{b_1}{b_n - b_1} \int_S \mathbb{E}[X \mid S = s] \, d\rho^*(s) \leq \bar{b} - \frac{b_1}{b_n - b_1}.$$

**Proof of Proposition 3.** The proof follows by simple algebra using that $X^\beta(s) = \Gamma(s) = \mathbb{E}^{\beta^*}[X \mid S = s]$, the characterization of the equilibrium return distribution obtained in Theorem 2 and the definition of $\phi_b$ from Eq. 4

$$SD^{\beta^*}[X, s]^2 = \mathbb{E}^{\beta^*}[X^2 \mid S = s] - \mathbb{E}^{\beta^*}[X \mid S = s]^2 = \int_0^1 \left( F_b^{-1}(y, s) \right)^2 \, dy - \Gamma(s)^2$$

$$= \int_0^1 \left[ \phi_b(y, s) \frac{b_n - b_1}{\Gamma(s)} \right]^2 \, dy - \Gamma(s)^2 = \Gamma(s)^2 \left( \int_0^1 \left[ \phi_b(y, s) \frac{b_n - b_1}{\Gamma(s)} \right]^2 \, dy - 1 \right)$$

$$= X^\beta(s)^2 \left( \sum_{k=1}^n \frac{b_k - b_1}{b - b_1} \left( \frac{n - 1}{k - 1} \right) y^{k-1} (1 - y)^{n-k} \right)^2 \, dy - 1 \right).$$

**Proof of Proposition 2.** Consider a trading strategy, which yields a final portfolio value $X$ in every state of the world $S$ equal to the expected value of the contests outcome, i.e.

$$X = \frac{d\rho(S)}{d\rho^*(S)} \text{ a.s.}.$$

First there exists a trading strategy leading to this portfolio outcome as in every state the expected value equals exactly the expected value of the contests portfolio. Furthermore as $X^i_T$ is a mean preserving spread to $Z$ every risk-averse agent prefers $Z$ over $X$. □
Proof of Proposition 6. If there exists a strategy such that the wealth at time \( t \) is given by

\[
X_t = S_T^{\frac{\mu - r}{\sigma^2}} \exp \left( \frac{1}{2} \left( \mu + r \right) \left( 1 - \frac{\mu - r}{\sigma^2} \right) T \right)
\]

than by construction \( X_T = \Gamma(S_T) \) and hence this strategy is a Kelly strategy. Applying Ito’s Lemma to the process \((X_t)_{t \in [0,T]}\) defined above yields that

\[
dX_t = X_t \left\{ \frac{\mu - r}{\sigma^2} (\mu dt + \sigma dW_t) + \mu - r \left[ \frac{\mu - r}{\sigma^2} - 1 \right] \frac{\sigma^2}{2} + \frac{1}{2} (\mu + r) \left( 1 - \frac{\mu - r}{\sigma^2} \right) \right\} \\
= X_t \left\{ \frac{\mu - r}{\sigma^2} dS_t + \left( 1 - \frac{\mu - r}{\sigma^2} \right) \left[ \frac{1}{2} (\mu + r) - \frac{\mu}{\sigma^2} \right] \right\}
\]

Thus, the final payoff \( X_T = \Gamma(S_T) \) is replicated by the trading strategy \( \beta_t = \frac{\mu - r}{\sigma^2} \) which is thus a Kelly strategy. \( \square \)

Proof of Proposition 5. First, note that the inverse cumulative distribution function for the equilibrium distribution \( F_b \) under the compensation structure \( b \in \mathbb{R}^n \) is given by

\[
F_b^{-1}(y, s) = \phi_b(y) \bar{x}_b(s) = \phi_b(y) \Gamma(s) \frac{b_n - b_1}{b - b_1}.
\]

Hence,

\[
(15) \quad \int_0^\alpha F_b^{-1}(y, s) dy \geq \int_0^\alpha F_b^{-1}(y, s) dy
\]

if and only if Eq. 14 is satisfied. As Eq. 15 is satisfied for all \( \alpha \in [0,1] \) if and only if \( F_b \) second order stochastically dominates \( F_b \) and the condition is independent of \( s \) we have shown the result. \( \square \)

Proof of Proposition 4. We have that

\[
\text{SD}^{\beta^*}[X]^2 = \int_S \mathbb{E}^{\beta^*} [X^2 | S = s] d\rho(s) - \mathbb{E}^{\beta^*} [X]^2
\]

\[
= \int_S \Gamma(s) \int_0^1 \left[ \sum_{k=1}^n \frac{b_k - b_1}{b - b_1} \left( \frac{n-1}{k-1} \right) y^{k-1}(1-y)^{n-k} \right]^2 dy d\rho(s) - \mathbb{E}^{\beta} [X]^2
\]

\[
= (R_b^2 + 1) \int_S \Gamma(s)^2 d\rho(s) - \mathbb{E}^{\beta} [X]^2
\]

\[
= (R_b^2 + 1) \left( \text{SD}^{\beta}[X]^2 + \mathbb{E}^{\beta} [X]^2 \right) - \mathbb{E}^{\beta} [X]^2
\]

\[
= (R_b^2 + 1) \text{SD}^{\beta}[X]^2 + R_b^2 \mathbb{E}^{\beta} [X].
\]
\textbf{Proof of Theorem 4}. Fix a vector of prices \( b \in \mathbb{R}^n \). When we add an agent we add a compensation of \( b_1 \) to the new compensation vector \( \tilde{b} \in \mathbb{R}^{n+1} \), i.e. \( \tilde{b}_k = b_{\max(k-1,1)} \) for all \( k \in \{1, \ldots, n+1\} \). As \( \tilde{b} = \tilde{b} \) and \( \tilde{b}_1 = b_1 \) we can without loss of generality assume \( b_1 = 0 \) and \( \bar{b} = 1 \) to simplify notation. We have that

\[
F^{-1}_b(y, s) = \pi_b(s) \phi_b(y) = \Gamma(s) \sum_{k=1}^{n} b_k - b_1 \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}
\]

Similarly, we have that

\[
F^{-1}_\tilde{b}(y, s) = \pi_b(s) \phi_{\tilde{b}}(y) = \Gamma(s) \sum_{k=1}^{n+1} \tilde{b}_k - \tilde{b}_1 \binom{n}{k-1} y^{k-1}(1-y)^{n+1-k}
\]

Thus, the difference between \( F^{-1}_b(y, s) \) and \( F^{-1}_\tilde{b}(y, s) \) is given by

\[
F^{-1}_b(y, s) - F^{-1}_\tilde{b}(y, s) = \Gamma(s) \left\{ \sum_{k=1}^{n} b_k \left[ \binom{n}{k} y^{k-1}(1-y)^{n-k} - \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} \right] \right\}
\]

\[
= \Gamma(s) \left\{ \sum_{k=1}^{n} b_k \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} \left[ \frac{n}{k} y - 1 \right] \right\}
\]

\[
= \Gamma(s) \left\{ \sum_{k=1}^{n} b_k q_k(y) \right\},
\]

where \( q_k(y) \triangleq \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} \left[ \frac{n}{k} y - 1 \right] \). We are going to prove that \( y \mapsto F^{-1}_b(y, s) - F^{-1}_\tilde{b}(y, s) \) satisfies the single crossing property, i.e. is first negative and then positive. To do this we use a result from Quah and Strulovici [2012]. We first prove that \( q_k \) and \( q_j \) satisfy obey signed ratio monotonicity (Definition 1 in Quah and Strulovici [2012]). Without loss of generality let \( k > j \). We have that \( q_k(y) < 0 \) and \( q_j(y) > 0 \) is equivalent to \( y < \frac{k}{n} \) and \( y > \frac{j}{n} \) and hence \( y \in \left( \frac{j}{n}, \frac{k}{n} \right) \). We get that

\[
- \frac{q_k(y)}{q_j(y)} = - \binom{n-1}{k-1} y^{k-1}(1-y)^{n-k} \left[ \frac{n}{k} y - 1 \right] = \left[ \frac{y}{1-y} \right]^{k-j} \binom{n-1}{j-1} \frac{1 - \frac{n}{j} y}{\frac{n}{j} y - 1}.
\]

\footnote{We say that \( f \) and \( g \) obey signed-ratio monotonicity if they satisfy the following conditions: (a) at any \( y \), such that \( g(y) < 0 \) and \( f(y) > 0 \), we have \( -\frac{g(y')}{f(y')} \geq -\frac{g(y'')}{f(y'')} \) for all \( y'' < y' \) and (b) at any \( y' \), such that \( g(y') > 0 \) and \( f(y') < 0 \), we have \( -\frac{g(s')}{f(s')} \leq -\frac{g(s'')}{f(s'')} \) for all \( s'' > s' \).}
To show that \(-\frac{q_k(y)}{q_j(y)}\) is decreasing for \(y \in (\frac{j}{n}, \frac{k}{n})\) we calculate the derivative of the logarithm

\[
\frac{\partial}{\partial y} \log \left( -\frac{q_k(y)}{q_j(y)} \right) = \frac{1}{1-y} - \frac{1}{k-y} + \frac{1}{y} - \frac{1}{y - \frac{j}{n}} < 0.
\]

As \(q_k(y) > 0\) implies \(y > \frac{k}{n}\) and \(q_j(y) < 0\) implies \(y < \frac{j}{n}\) we have that \(\{y: q_k(y) > 0 \text{ and } q_j(y) < 0\} = \emptyset\) and hence \(q_k\) and \(q_j\) obey signed ratio monotonicity. By Theorem 1 in Quah and Strulovici [2012] and as \(b_k \geq 0\) for all \(k\) it follows that \(F^{-1}_b(y, s) - F^{-1}_b(y, s)\) has the single crossing property for every \(s \in S\).

As

\[
\int_0^1 F^{-1}_b(y, s) dy = \Gamma(s) = \int_0^1 F^{-1}_b(y, s) dy
\]

it follows that for all \(\alpha \in [0, 1]\)

\[
\int_0^\alpha F^{-1}_b(y, s) - F^{-1}_b(y, s) \, dy \leq 0
\]

and hence \(F_b(\cdot, s)\) second order stochastically dominates \(F_b(\cdot, s)\).

\(\square\)

**Proof of Theorem 3.** We show that shifting mass from the \(k\)-th to \((k+1)\)-th price is bad for risk-averse investors. Let

\[
\tilde{b}_j = b_j
\]

for all \(j \notin \{k, k+1\}\) and \(k > 1\). Without loss of generality restrict attention to normalized price vectors \(b, \tilde{b}\). It suffices to prove single crossing

\[
F^{-1}_b(y, s) - F^{-1}_\tilde{b}(y, s) = \pi_b(s)\phi_b(y) - \pi_\tilde{b}(s)\phi_\tilde{b}(y) = \Gamma(s) \sum_{k=1}^n \frac{b_k - \tilde{b}_k}{b_k - b_1} \left(\frac{n-1}{k-1}\right) y^{k-1}(1-y)^{n-k}.
\]

Here we used that \(k > 1\) and thus \(b_1 = \tilde{b}_1\) and as we are shifting compensation while not changing the sum of compensations \(\tilde{b} = \tilde{b}\). Increasing the \((k+1)\)-th compensation \(\tilde{b}_{k+1} = b_{k+1} + \epsilon\) and decreasing the \(k\)-th price \(\tilde{b}_k = b_k - \epsilon\) yields

\[
F^{-1}_b(y, s) - F^{-1}_\tilde{b}(y, s) = \Gamma(s) \epsilon \left[ \left(\frac{n-1}{k}\right) y^{k-1}(1-y)^{n-k-1} - \left(\frac{n-1}{k-1}\right) y^{k-1}(1-y)^{n-k} \right]
\]

\[
\quad = \Gamma(s) \epsilon y^{k-1}(1-y)^{n-k-1} \left(\frac{n-1}{k-1}\right) \left[ \frac{k}{n-k} y - (1-y) \right]
\]

\[
\quad = \Gamma(s) \epsilon y^{k-1}(1-y)^{n-k-1} \left(\frac{n-1}{k-1}\right) \left[ \frac{n}{n-k} y - 1 \right].
\]

As the last term is linear in \(y\) the difference in inverse cdfs \(F^{-1}_b(y, s) - F^{-1}_\tilde{b}(y, s)\) is changing its sign at most once from positive to negative. Hence \(F_b(y) - F_\tilde{b}(y)\) is changing sign at most once.
from negative to positive and as $F_b$ and $\tilde{F}_b$ have the same mean we have that for all $\alpha \geq 0$
\[ \int_0^\alpha F_b(y) - \tilde{F}_b(y) dy \leq 0. \]

Consequently, $F_b$ dominates $\tilde{F}_b$ in the sense of second order stochastic dominance. For shifts of the $k$-th to the $j$-th prize the result follows from iteratively shifting mass to the next higher prize.28

Proof of Proposition 7. Denote by $F^* : \mathbb{R}_+ \rightarrow [0, 1]$ the cdf of the symmetric equilibrium bid distribution. Let $[x, \bar{x}]$ convex hull of the equilibrium scores

$$[x, \bar{x}] \triangleq [\min \text{ supp}(F^*), \max \text{ supp}(F^*)]$$

It follows from the standard logic in static game theory with a continuous action space (see, e.g., Burdett and Judd [1983]) that $x = 0$ and $F^*$ is increasing and continuous on the support $\text{supp}(F) = [0, \bar{x}]$. Let us denote by $\pi(x) = \mathbb{E}[b_{R(i)} | X' = x]$ the expected price the agent receives when making a bid of $x$. By the same argument as in section 3

$$\pi(x) = b_1 + (b_n - b_1) \phi_b(F^*(x)).$$

As each bid in $(0, \bar{x})$ is in the support of the bid distribution and the expected payoff in any symmetric equilibrium equals the payoff when an agent makes a bid of zero $b_1$ the expected payoff must be constant equal to $b_1$ on the support, i.e. for all $x \in [0, \bar{x}]$

$$b_1 = \pi(x) - cx = [b_1 + (b_n - b_1) \phi_b(F(x))] - cx$$

$$\Rightarrow F_b(x) = \phi_b^{-1}\left(\frac{cx}{b_n - b_1}\right).$$

As $\phi_b(1) = 1$ we thus have that the right end-point of the bid distribution is given by $\bar{x} = \frac{b_n - b_1}{c}$. The inverse distribution function is given by

$$(F_b)^{-1}(y) = \frac{(b_n - b_1) \phi_b(y)}{c}.$$  

Consequently, the expected equilibrium bid is given by

$$\mathbb{E}^{F_b}[X_i] = \int_0^1 (F_b)^{-1}(y) dy = \frac{1}{c} \left\{ (b_n - b_1) \int_0^1 \phi_b(y) dy \right\} = \frac{b_b - b_1}{c}. \quad \square$$

Proof of Proposition 9. Consider an arbitrary symmetric equilibrium distribution $F_b : \mathbb{R}_+ \times S \mapsto [0, 1]$. By the Lagrangian principle there exists a Lagrange multiplier $\lambda \geq 0$ such that

$$F_b \in \arg \max_G \int_S \int_{\mathbb{R}_+} \phi_b(F_b(x, s)) dG(x, s) d\rho(s) - \lambda \int_S \int_{\mathbb{R}_+} x dG(x, s) d\rho^*(s) \quad (16)$$

28Note, that the result is correct even if $b_{k+1} = b_{k+2}$ as the above argument shows that the shift from the $k$-th to the $k + 1$-th prize increases the riskiness of the distribution in the sense of second order stochastic dominance even if the resulting compensation vector is not ordered.
over the set of all distributions $G : \mathbb{R}_+ \times S \to [0, 1]$. Using that $\int \cdot \Gamma(s)^{-1} d\rho(s) = \int \cdot d\rho^*(s)$ we get that (16) equals
\[
\int_S \left[ \phi_b(F_b(x, s)) - \lambda \Gamma(s)^{-1} x \right] dG(x, s) d\rho(s).
\]
Note, that this problem is completely separable over the state $s$, i.e. a distribution $G(x, s)$ is an equilibrium distribution if and only if for every state $s$ it places mass only on points $x$ that maximize the expected payoff in this state given by
\[
(17) \quad \phi_b(G(x, s)) - \lambda \Gamma(s)^{-1} x.
\]
Eq. (17) equals the expected payoff in symmetric equilibrium of an all-pay contest with linear cost $c = \lambda \Gamma(s)^{-1}$ and compensation structure $b$ where $x \in \mathbb{R}_+$ is the bid of an agent. As shown in 7 the expected bid in this all-pay contest equals $\frac{c}{b-b_1} = \frac{\lambda \Gamma(s)^{-1}}{b-b_1}$. Feasibility, of the resulting equilibrium distribution implies that
\[
1 = \int_S \mathbb{E}[X \mid S = s] d\rho^*(s) = \int_S \frac{\lambda \Gamma(s)^{-1}}{b - b_1} d\rho^*(s) \Rightarrow \lambda = \frac{b - b_1}{\int_S \Gamma(s)^{-1} d\rho^*(s)}.
\]
TABLE 2. Expected return and standard deviation of the return for different investments assuming fifty competing fund-managers and the compensation structure estimated in Reuter and Zitzewitz [2010], assuming a Black-Scholes model with $T = 1$ and different parameters.

<table>
<thead>
<tr>
<th>Bond Risk Asset Funds Strategy Index Strategy</th>
<th>Return</th>
<th>Return StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_i$</td>
<td>Return</td>
<td>StdDev</td>
</tr>
<tr>
<td>2%</td>
<td>2%</td>
<td>2%</td>
</tr>
<tr>
<td>7%</td>
<td>6%</td>
<td>5%</td>
</tr>
<tr>
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TABLE 3. Expected return and standard deviation of the return for different investments assuming fifty competing fund-managers and the compensation structure $\beta_i = (0, ..., 0, 1)$, assuming a Black-Scholes model with $T = 1$ and different parameters.

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<th>Return</th>
<th>Return StDev</th>
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Table 4. Expected return and standard deviation of the return for different investments assuming fifty competing fund-managers and the compensation structure $b = (0, 1, \ldots, 1)$, assuming a Black-Scholes model with $T = 1$ and different parameters.